Week 7 (first part)

Linear transformations between vector spaces:

Definition 4.26: Let V, W be two vector spaces. A function $T : V \to W$ is called a *linear transformation (between vector spaces)* if the following linearity axiom holds for all $\mathbf{x}_1, \mathbf{x}_2 \in V$ and all $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$\underbrace{T(\underbrace{\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2}_{\in V, \text{Lemma 4.16}})}_{\in W} = \underbrace{\lambda_1 \underbrace{T(\mathbf{x}_1)}_{\in W} + \lambda_2 \underbrace{T(\mathbf{x}_2)}_{\in W}}_{\in W, \text{Lemma 4.16}}.$$

So far: $V = \mathbb{R}^n, W = \mathbb{R}^m$. Different example: Matrix "flattening":

$$V = \mathbb{R}^{2\times 2}, \quad W = R^4, \quad T : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

This is *bijective* (undoable).

Lemma 4.27: Let $T : V \to W$ be a bijective linear transformation, $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\} \subseteq V$, $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_\ell)\} \subseteq W$. Then

- $|T(B)| = |B|$.
- B is a basis of $V \Leftrightarrow T(B)$ is a basis of W. We therefore also have $\dim(V) = \dim(W)$.

Definition 4.28: If there is a bijective linear transformation $T : V \to W$, then V and W are called *isomorphic*, and T is called an *isomorphism* between V and W.

Isomorphism examples:

- $V = \mathbb{R}^{m\times n}$, $W = \mathbb{R}^{nm}$, T the "flattening" transformation
- $V =$ set of degree-2 polynomials, $W = \mathbb{R}^3$, $T : p_0 + p_1 x + p_2 x^2 \mapsto \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}$

Computing (bases of) the three fundamental subspaces (Section 4.3)

Column space $\mathbf{C}(A)$, row space $\mathbf{R}(A)$ and nullspace $\mathbf{N}(A)$ of a matrix A.

Based on $A \to R$ (Gauss-Jordan):

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column space:

Theorem 4.31: If R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$, then A has its independent columns at indices $j_1, j_2, \ldots, j_r$, and these columns form a basis of the column space $\mathbf{C}(A)$. In particular,

$$\dim(\mathbf{C}(A)) = \mathbf{rank}(A) = r.$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \rightarrow \quad R = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} j_1 & & j_2 & \end{array} \end{array}$$

Proof. Independent columns of A: basis of $\mathbf{C}(A)$ (Lemma 4.19), and R reveals their indices $j_1, j_2, \ldots, j_r$ (Theorem 3.18). Thus $\dim(\mathbf{C}(A)) = r$ (Definition 4.25) and $\mathbf{rank}(A) = r$ (Definition 2.10). □

Row space:

Theorem 4.32: If R in $\mathrm{RREF}(j_1, j_2, \ldots, j_r)$, the first r rows of R form a basis of the row space $\mathbf{R}(A)$. In particular,

$$\dim(\mathbf{R}(A)) = r.$$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \quad \rightarrow \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof. $R = MA$ with M invertible (Theorem 3.17) $\Rightarrow$ R and A have the same row space (Lemma 3.5). The r nonzero rows of R already span the row space, and they are linearly independent (private nonzeros at the downward steps). So the first r rows are a basis and $\dim(\mathbf{R}(A)) = r$ (Definition 4.25). □

Theorem 4.33: Let A be an $m \times n$ matrix. Then $\mathbf{rank}(A) = \mathbf{rank}(A^\top)$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} \qquad A^\top = \begin{array}{l} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \\ \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3 = 2\mathbf{v}_2 - \mathbf{v}_1 \end{array}$$

Proof.

- $\mathbf{rank}(A) = \dim(\mathbf{C}(A)) = r$ (Theorem 4.31)
- $\mathbf{rank}(A^\top) = \dim(\mathbf{C}(A^\top)) \underbrace{=}_{\mathbf{R}(A)=\mathbf{C}(A^\top)} \dim(\mathbf{R}(A)) = r$ (Theorem 4.32)

□

Corollary 4.34: Let A be an $m \times n$ matrix of rank r. Then $r \leq \min(n, m)$.

Proof. $r = \mathbf{rank}(A) \leq n$ (number of columns), $r = \mathbf{rank}(A^\top) \leq m$ (number of rows). □

Nullspace:

$$A = \begin{bmatrix}1 & 2 & 0 & 3\\ 2 & 4 & 1 & 4\\ 3 & 6 & 2 & 5\end{bmatrix} \quad\to\quad R = \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} \quad\to\quad R' = \begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & -2\end{bmatrix}$$

Lemma 3.3: $\mathbf{N}(A) = \mathbf{N}(R)(= \mathbf{N}(R'))$.

Dimension / basis by example: $\mathbf{x} \in \mathbf{N}(R') \Leftrightarrow R'\mathbf{x} = \mathbf{0}$, and this can be written as

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \underbrace{\begin{pmatrix}x_1\\ x_3\end{pmatrix}}_{\in \mathbb{R}^r} + \begin{bmatrix}2 & 3\\ 0 & -2\end{bmatrix} \underbrace{\begin{pmatrix}x_2\\ x_4\end{pmatrix}}_{\in \mathbb{R}^{n-r}} = \mathbf{0}. \qquad \text{Or:} \quad \begin{pmatrix}x_1\\ x_3\end{pmatrix} = -\begin{bmatrix}2 & 3\\ 0 & -2\end{bmatrix}\begin{pmatrix}x_2\\ x_4\end{pmatrix}. \qquad (*)$$

$\Rightarrow \mathbf{N}(R')$ is isomorphic to $\mathbb{R}^{n-r}$ and therefore has dimension $n - r$ (Lemma 4.27):

$$T : \underbrace{\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}}_{\in \mathbf{N}(R')} \mapsto \underbrace{\begin{pmatrix}x_2\\ x_4\end{pmatrix}}_{\in \mathbb{R}^{n-r}}$$

is a bijective linear transformation:

"for every possible output $\begin{pmatrix}x_2\\ x_4\end{pmatrix} \in \mathbb{R}^{n-r}$, exactly one input $\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} \in \mathbf{N}(R')$ leads to it."

We get the missing input values x_1, x_3 from $(*)$.

Basis of $\mathbf{N}(A) = \mathbf{N}(R')$: inputs that lead to outputs $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-r}$ (Theorem 4.36).

Summary:

$$\begin{array}{ccccccc}
 & & \mathbf{C}(A)\ (\dim = r) & & \mathbf{N}(A)\ (\dim = n-r) & & \mathbf{R}(A)\ (\dim = r)\\
 & & & & \overbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}^{x_2,x_4}\ \overbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}^{x_2,x_4} & & \\
\text{Bases:} & & \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}\ \begin{pmatrix}0\\ 1\\ 2\end{pmatrix} & & \overbrace{\begin{pmatrix}-2\\ 0\end{pmatrix}}^{x_1,x_3}\ \overbrace{\begin{pmatrix}-3\\ 2\end{pmatrix}}^{x_1,x_3} & & \begin{pmatrix}1\\ 2\\ 0\\ 3\end{pmatrix}\ \begin{pmatrix}0\\ 0\\ 1\\ -2\end{pmatrix}\\
 & & \uparrow\quad\uparrow & & \uparrow\quad\uparrow & & \uparrow\quad\uparrow\\
\underbrace{\begin{bmatrix}1 & 0\\ 2 & 1\\ 3 & 2\end{bmatrix}}_{C} & \leftarrow & \underbrace{\begin{bmatrix}1 & 2 & 0 & 3\\ 2 & 4 & 1 & 4\\ 3 & 6 & 2 & 5\end{bmatrix}}_{A=CR'} & \to & \underbrace{\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}}_{R \text{ in RREF(1,3)}} & \to & \underbrace{\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & -2\end{bmatrix}}_{R' \text{ in RREF(1,3)}}\\
 & & \text{Theorem 4.31} & & \text{Theorem 4.36} & & \text{Theorem 4.32}
\end{array}$$

All solutions of $A\mathbf{x} = \mathbf{b}$ (Section 4.4)

Definition 4.37: Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. The set

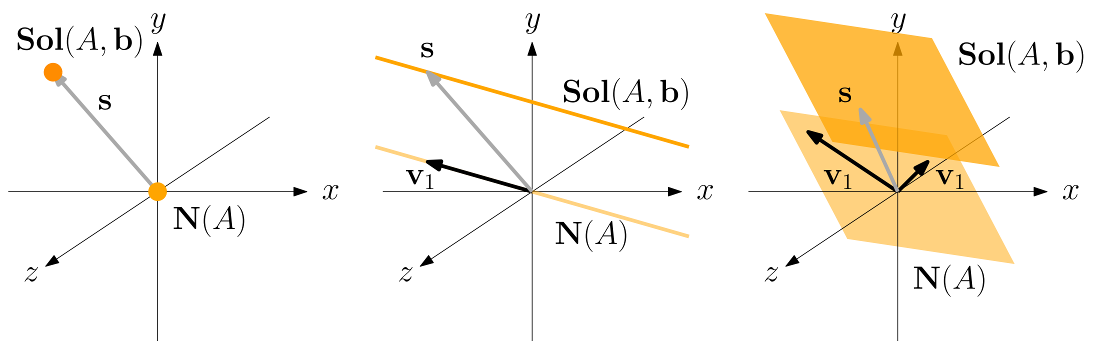
$$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^n$$

is the *solution space* of $A\mathbf{x} = \mathbf{b}$.

Not a subspace if $\mathbf{b} \neq \mathbf{0}$, because then $\mathbf{0} \notin \mathbf{Sol}(A, \mathbf{b})$ (see Lemma 4.9).

$\mathbf{Sol}(A, \mathbf{0}) = \mathbf{N}(A)$.

$\mathbf{Sol}(A, \mathbf{b}) =$ "shifted nullspace":
Theorem 4.38: Let $\mathbf{s}$ be some solution of $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{s} + \mathbf{x} : \mathbf{x} \in \mathbf{N}(A)\}$.



Proof.
(i) for every $\mathbf{y} \in \mathbf{Sol}(A, \mathbf{b})$, there is $\mathbf{x} \in \mathbf{N}(A)$ with $\mathbf{y} = \mathbf{s} + \mathbf{x}$. To see this, write

$$\mathbf{y} = \mathbf{s} + \underbrace{(\mathbf{y} - \mathbf{s})}_{\mathbf{x}}.$$

Then $A\mathbf{x} = A(\mathbf{y} - \mathbf{s}) = A\mathbf{y} - A\mathbf{s} = \mathbf{b} - \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{x} \in \mathbf{N}(A)$.
(ii) for every vector $\mathbf{y} = \mathbf{s} + \mathbf{x}$ with $\mathbf{x} \in \mathbf{N}(A)$ we have $\mathbf{y} \in \mathbf{Sol}(A, \mathbf{b})$. To see this:

$$A\mathbf{y} = A(\mathbf{s} + \mathbf{x}) = A\mathbf{s} + A\mathbf{x} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

□

Theorem 4.39: Let A be an $m \times n$ matrix of rank r. If $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{Sol}(A, b)$ has dimension $n - r$, where $\dim(\mathbf{Sol}(A, b)) := \dim(\mathbf{N}(A))$.

Computing $\mathbf{Sol}(A, \mathbf{b})$: compute some solution $\mathbf{s}$ (Gauss-Jordan, Theorem 3.20) and a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-r}\}$ of $\mathbf{N}(A)$ (Theorem 4.36). Then

$$\mathbf{Sol}(A, \mathbf{b}) = \left\{ \mathbf{s} + \sum_{i=1}^{n-r} \lambda_i \mathbf{v}_i : \lambda_i \in \mathbb{R} \text{ for } i \in [n - r] \right\}.$$