

Lineare Algebra

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Gauss Elimination, Worked Example

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Solve a system of equations with Gauss elimination and back substitution

System of linear equations

$$\begin{array}{rrcr} 1x_1 & +2x_2 & +3x_3 & +4x_4 = -1 \\ 5x_1 & +6x_2 & +7x_3 & +8x_4 = 5 \\ 9x_1 & +10x_2 & +11x_3 & +12x_4 = 4 \\ 13x_1 & +14x_2 & +15x_3 & +16x_4 = 2 \end{array}$$

Matrix vector form $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \end{bmatrix}}_{\mathbf{b}}$$

Algorithm:

- ▶ Turn the red entries into 0 using row subtractions and row exchanges!
- ▶ Use back substitution to solve for $\mathbf{x}$!

Elimination in column 1: Subtract 5 · row 1 from row 2

$$\begin{array}{c} A \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -1 \\ 5 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\text{row 2 : } \quad 5 \quad 6 \quad 7 \quad 8 \quad 5$$

$$\begin{array}{c} E_{21} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array} \right] \end{array} \quad \begin{array}{c} E_{21} \cdot \downarrow \\ \left[\begin{array}{c} -1 \\ 10 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\begin{array}{l} 5 \cdot \text{row 1 :} \\ \hline = \text{row 2'} : \end{array} \quad \begin{array}{ccccc} 5 & 10 & 15 & 20 & -5 \\ 0 & -4 & -8 & -12 & 10 \end{array}$$

$$\text{elimination matrix: } E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract 9 · row 1 from row 3

$$\begin{array}{c} A \qquad \qquad \mathbf{b} \\ \left[\begin{array}{cccc} \mathbf{1} & 2 & 3 & 4 \\ \mathbf{0} & -4 & -8 & -12 \\ \mathbf{9} & \mathbf{10} & 11 & 12 \\ \mathbf{13} & \mathbf{14} & \mathbf{15} & 16 \end{array} \right] \quad \left[\begin{array}{c} -1 \\ 10 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\text{row 3 : } \quad \mathbf{9} \quad \mathbf{10} \quad 11 \quad 12 \quad 4$$

$$\begin{array}{c} E_{31} \cdot \downarrow \\ \left[\begin{array}{cccc} \mathbf{1} & 2 & 3 & 4 \\ \mathbf{0} & -4 & -8 & -12 \\ \mathbf{0} & \mathbf{-8} & -16 & -24 \\ \mathbf{13} & \mathbf{14} & \mathbf{15} & 16 \end{array} \right] \quad \left[\begin{array}{c} -1 \\ 10 \\ 13 \\ 2 \end{array} \right] \end{array}$$

$$\frac{9 \cdot \text{row 1 :}}{\quad} \quad \frac{9}{\quad} \quad \frac{18}{\quad} \quad \frac{27}{\quad} \quad \frac{36}{\quad} \quad \frac{-9}{\quad}$$

$$= \text{row 3'} : \quad \mathbf{0} \quad \mathbf{-8} \quad -16 \quad -24 \quad 13$$

$$\text{elimination matrix: } E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -9 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract $13 \cdot$ row 1 from row 4

$$\begin{array}{c} A \\ \left[\begin{array}{cccc} \mathbf{1} & 2 & 3 & 4 \\ \mathbf{0} & -4 & -8 & -12 \\ \mathbf{0} & -8 & -16 & -24 \\ \mathbf{13} & 14 & 15 & 16 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -1 \\ 10 \\ 13 \\ 2 \end{array} \right] \end{array}$$

$$\begin{array}{c} E_{41} \cdot \downarrow \\ \left[\begin{array}{cccc} \mathbf{1} & 2 & 3 & 4 \\ \mathbf{0} & -4 & -8 & -12 \\ \mathbf{0} & -8 & -16 & -24 \\ \mathbf{0} & -12 & -24 & -36 \end{array} \right] \end{array} \quad \begin{array}{c} E_{41} \cdot \downarrow \\ \left[\begin{array}{c} -1 \\ 10 \\ 13 \\ 15 \end{array} \right] \end{array}$$

$$\text{row 4 : } \mathbf{13} \quad \mathbf{14} \quad \mathbf{15} \quad 16 \quad 2$$

$$\begin{array}{r} 13 \cdot \text{row 1 :} \quad \underline{13} \quad \underline{26} \quad \underline{39} \quad \underline{52} \quad \underline{-13} \end{array}$$

$$= \text{row 4'} : \quad \mathbf{0} \quad \mathbf{-12} \quad \mathbf{-24} \quad -36 \quad 15$$

$$\text{elimination matrix: } E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -13 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 2: Subtract 2 · row 2 from row 3

$$\begin{array}{c} A \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -1 \\ 10 \\ 13 \\ 15 \end{array} \right] \end{array}$$

$$\text{row 3 : } \quad 0 \quad -8 \quad -16 \quad -24 \quad 13$$

$$\begin{array}{c} E_{32} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & -12 & -24 & -36 \end{array} \right] \end{array} \quad \begin{array}{c} E_{32} \cdot \downarrow \\ \left[\begin{array}{c} -1 \\ 10 \\ -7 \\ 15 \end{array} \right] \end{array}$$

$$\begin{array}{rcl} 2 \cdot \text{row 2 :} & 0 & -8 \quad -16 \quad -24 \quad 20 \\ \hline = \text{row 3'} : & 0 & 0 \quad 0 \quad 0 \quad -7 \end{array}$$

$$\text{elimination matrix: } E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 2: Subtract 3 · row 2 from row 4

$$\begin{array}{c} A \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & -12 & -24 & -36 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -1 \\ 10 \\ -7 \\ 15 \end{array} \right] \end{array}$$

$$\text{row 4 : } \quad 0 \quad -12 \quad -24 \quad -36 \quad 15$$

$$\begin{array}{c} E_{42} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array} \quad \begin{array}{c} E_{42} \cdot \downarrow \\ \left[\begin{array}{c} -1 \\ 10 \\ -7 \\ -15 \end{array} \right] \end{array}$$

$$\begin{array}{r} 3 \cdot \text{row 2 :} \quad 0 \quad -12 \quad -24 \quad -36 \quad 30 \\ \hline \end{array}$$

$$= \text{row 4'} : \quad 0 \quad 0 \quad 0 \quad 0 \quad -15$$

$$\text{elimination matrix: } E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

No row exchange possible: Give up!

$$\begin{array}{c} A \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{bmatrix} \begin{array}{c} \mathbf{b} \end{array} \begin{bmatrix} -1 \\ 10 \\ -7 \\ -15 \end{bmatrix}$$