

# **Lineare Algebra**

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Gauss Elimination, Worked Example

Bernd Gärtner

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## Solve a system of equations with Gauss elimination and back substitution

System of linear equations

$$\begin{array}{rrcrcl} & 3x_2 & -1x_3 & +2x_4 & = & -1 \\ 2x_1 & +1x_2 & -1x_3 & +2x_4 & = & 5 \\ -2x_1 & +5x_2 & -2x_3 & +6x_4 & = & 4 \\ 4x_1 & +11x_2 & -4x_3 & +8x_4 & = & 2 \end{array}$$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 0 & 3 & -1 & 2 \\ 2 & 1 & -1 & 2 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \end{bmatrix}}_{\mathbf{b}}$$

Algorithm:

- ▶ Turn the red entries into 0 using row subtractions and row exchanges!
- ▶ Use back substitution to solve for  $\mathbf{x}$ !

## Zero pivot: Exchange rows 1 and 2

$$\begin{array}{cc} \begin{array}{c} A \\ \left[ \begin{array}{cccc} \mathbf{0} & 3 & -1 & 2 \\ \color{red}{2} & 1 & -1 & 2 \\ \color{red}{-2} & \color{red}{5} & -2 & 6 \\ \color{red}{4} & \color{red}{11} & \color{red}{-4} & 8 \end{array} \right] \end{array} & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} -1 \\ 5 \\ 4 \\ 2 \end{array} \right] \end{array} & \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \end{array} \\ \\ \begin{array}{c} P_{12} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \color{red}{0} & 3 & -1 & 2 \\ \color{red}{-2} & \color{red}{5} & -2 & 6 \\ \color{red}{4} & \color{red}{11} & \color{red}{-4} & 8 \end{array} \right] \end{array} & \begin{array}{c} P_{12} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array} & \end{array}$$

permutation matrix:  $P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Elimination in column 1: Subtract  $0 \cdot$  row 1 from row 2

$$\begin{array}{c} A \qquad \qquad \mathbf{b} \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \mathbf{0} & 3 & -1 & 2 \\ -2 & \mathbf{5} & -2 & 6 \\ 4 & \mathbf{11} & -4 & 8 \end{array} \right] \quad \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\text{row 2 : } \mathbf{0} \quad 3 \quad -1 \quad 2 \quad -1$$

$$\begin{array}{c} E_{21} \cdot \downarrow \qquad \qquad E_{21} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \mathbf{0} & 3 & -1 & 2 \\ -2 & \mathbf{5} & -2 & 6 \\ 4 & \mathbf{11} & -4 & 8 \end{array} \right] \quad \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\begin{array}{l} 0 \cdot \text{row 1 : } \quad \frac{0}{0} \quad \frac{0}{3} \quad \frac{0}{-1} \quad \frac{0}{2} \quad \frac{0}{-1} \\ \hline = \text{row 2'} : \quad \mathbf{0} \quad 3 \quad -1 \quad 2 \quad -1 \end{array}$$

$$\text{elimination matrix: } E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract  $(-1) \cdot \text{row 1}$  from row 3

$$\begin{array}{c} \mathbf{A} \qquad \qquad \mathbf{b} \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \mathbf{0} & 3 & -1 & 2 \\ \mathbf{-2} & \mathbf{5} & -2 & 6 \\ \mathbf{4} & \mathbf{11} & \mathbf{-4} & 8 \end{array} \right] \quad \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\text{row 3 :} \quad \mathbf{-2} \quad \mathbf{5} \quad -2 \quad 6 \quad 4$$

$$\begin{array}{c} E_{31} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \mathbf{0} & 3 & -1 & 2 \\ \mathbf{0} & \mathbf{6} & -3 & 8 \\ \mathbf{4} & \mathbf{11} & \mathbf{-4} & 8 \end{array} \right] \quad \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ 2 \end{array} \right] \end{array}$$

$$\frac{(-1) \cdot \text{row 1 :} \quad -2 \quad -1 \quad 1 \quad -2 \quad -5}{\phantom{= \text{row 3'} :}}$$

$$= \text{row 3'} : \quad \mathbf{0} \quad \mathbf{6} \quad -3 \quad 8 \quad 9$$

$$\text{elimination matrix: } E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract  $2 \cdot$  row 1 from row 4

$$\begin{array}{cc}
 \begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 6 & -3 & 8 \\ 4 & 11 & -4 & 8 \end{array} \right] \end{array} & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ 2 \end{array} \right] \end{array} \\
 E_{41} \cdot \downarrow & E_{41} \cdot \downarrow \\
 \begin{array}{c} \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 6 & -3 & 8 \\ 0 & 9 & -2 & 4 \end{array} \right] \end{array} & \begin{array}{c} \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ -8 \end{array} \right] \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 \text{row 4 : } \quad 4 \quad 11 \quad -4 \quad 8 \quad 2 \\
 \\
 \begin{array}{r}
 2 \cdot \text{row 1 : } \quad 4 \quad 2 \quad -2 \quad 4 \quad 10 \\
 \hline
 \\
 = \text{row 4'} : \quad 0 \quad 9 \quad -2 \quad 4 \quad -8
 \end{array}
 \end{array}$$

elimination matrix:  $E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

Elimination in column 2: Subtract  $2 \cdot$  row 2 from row 3

$$\begin{array}{c} A \qquad \qquad \mathbf{b} \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & \mathbf{6} & -3 & 8 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} \quad \begin{array}{c} \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ -8 \end{array} \right] \end{array}$$

$$\text{row 3 : } \quad 0 \quad \mathbf{6} \quad -3 \quad 8 \quad 9$$

$$\begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & \mathbf{0} & -1 & 4 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} \quad \begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -8 \end{array} \right] \end{array}$$

$$\begin{array}{r} 2 \cdot \text{row 2 :} \quad 0 \quad 6 \quad -2 \quad 4 \quad -2 \\ \hline = \text{row 3'} : \quad 0 \quad 0 \quad -1 \quad 4 \quad 11 \end{array}$$

$$\text{elimination matrix: } E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 2: Subtract  $3 \cdot$  row 2 from row 4

$$\begin{array}{cc}
 \begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -8 \end{array} \right] \end{array} \\
 E_{42} \cdot \downarrow & E_{42} \cdot \downarrow \\
 \begin{array}{c} \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right] \end{array} & \begin{array}{c} \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -5 \end{array} \right] \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 \text{row 4 : } \quad 0 \quad \mathbf{9} \quad -2 \quad 4 \quad -8 \\
 \\
 \begin{array}{c} 3 \cdot \text{row 2 : } \quad 0 \quad 9 \quad -3 \quad 6 \quad -3 \\ \hline
 \end{array} \\
 \\
 = \text{row 4' : } \quad 0 \quad 0 \quad \mathbf{1} \quad -2 \quad -5
 \end{array}$$

elimination matrix:  $E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$



Elimination in column 3: Subtract  $(-1) \cdot \text{row 3}$  from row 4

$$\begin{array}{cc}
 \begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right] \end{array} & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -5 \end{array} \right] \end{array} \\
 E_{43} \cdot \downarrow & E_{43} \cdot \downarrow \\
 \begin{array}{c} \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{array} \right] \end{array} & \begin{array}{c} \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ 6 \end{array} \right] \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 \text{row 4 : } \quad 0 \quad 0 \quad \mathbf{1} \quad -2 \quad -5 \\
 \\
 \begin{array}{l}
 (-1) \cdot \text{row 3 : } \quad 0 \quad 0 \quad 1 \quad -4 \quad -11 \\
 \hline
 = \text{row 4' : } \quad 0 \quad 0 \quad 0 \quad 2 \quad 6
 \end{array}
 \end{array}$$

elimination matrix:  $E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

## Back substitution to compute $x_4, x_3, x_2, x_1$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ -1 \\ 11 \\ 6 \end{bmatrix}}_{\mathbf{b}}$$

System of linear equations

$$\begin{array}{rrcrcl} 2x_1 & +1x_2 & -1x_3 & +2x_4 & = & 5 \\ & 3x_2 & -1x_3 & +2x_4 & = & -1 \\ & & -1x_3 & +4x_4 & = & 11 \\ & & & 2x_4 & = & 6 \end{array}$$

| eq. | before substitution             | after substitution | solution   |   |
|-----|---------------------------------|--------------------|------------|---|
| 1   | $2x_1 + 1x_2 - 1x_3 + 2x_4 = 5$ | $2x_1 + 3 = 5$     | $x_1 = 1$  | ↑ |
| 2   | $3x_2 - 1x_3 + 2x_4 = -1$       | $3x_2 + 5 = -1$    | $x_2 = -2$ |   |
| 3   | $-1x_3 + 4x_4 = 11$             | $-1x_3 + 12 = 11$  | $x_3 = 1$  |   |
| 4   | $2x_4 = 6$                      |                    | $x_4 = 3$  |   |