

Lineare Algebra

ETH Zürich, HS 2024, 401-0131-00L

Gauss Elimination, Worked Example

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Solve a system of equations with Gauss elimination and back substitution

System of linear equations

$$\begin{array}{rrcrcl} 1x_1 & +9x_2 & +2x_3 & +5x_4 & = & -2 \\ 7x_1 & -3x_2 & +2x_3 & & = & 8 \\ 6x_1 & +8x_2 & +1x_3 & -1x_4 & = & 2 \\ -2x_1 & +3x_2 & & -5x_4 & = & 3 \end{array}$$

Matrix vector form $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 1 & 9 & 2 & 5 \\ 7 & -3 & 2 & 0 \\ 6 & 8 & 1 & -1 \\ -2 & 3 & 0 & -5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 8 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

Algorithm:

- ▶ Turn the red entries into 0 using row subtractions and row exchanges!
- ▶ Use back substitution to solve for $\mathbf{x}$!

Elimination in column 1: Subtract $7 \cdot$ row 1 from row 2

$$\begin{array}{c} A \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{7} & -3 & 2 & 0 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -2 \\ 8 \\ 2 \\ 3 \end{array} \right] \end{array}$$

$$\text{row 2 : } \mathbf{7} \quad -3 \quad 2 \quad 0 \quad 8$$

$$\begin{array}{c} E_{21} \cdot \downarrow \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} E_{21} \cdot \downarrow \\ \left[\begin{array}{c} -2 \\ 22 \\ 2 \\ 3 \end{array} \right] \end{array}$$

$$\begin{array}{r} 7 \cdot \text{row 1 :} \quad 7 \quad 63 \quad 14 \quad 35 \quad -14 \\ \hline = \text{row 2'} : \quad \mathbf{0} \quad -66 \quad -12 \quad -35 \quad 22 \end{array}$$

$$\text{elimination matrix: } E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract 6 · row 1 from row 3

$$\begin{array}{cc} & \mathbf{A} \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] & \mathbf{b} \end{array}$$

$$\text{row 3 : } \mathbf{6} \quad \mathbf{8} \quad 1 \quad -1 \quad 2$$

$$\begin{array}{cc} & E_{31} \cdot \downarrow \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{0} & -46 & -11 & -31 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] & E_{31} \cdot \downarrow \end{array}$$

$$\begin{array}{rccccc} 6 \cdot \text{row 1 :} & 6 & 54 & 12 & 30 & -12 \\ \hline \end{array}$$

$$= \text{row 3'} : \quad \mathbf{0} \quad -46 \quad -11 \quad -31 \quad 14$$

$$\text{elimination matrix: } E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 1: Subtract $(-2) \cdot \text{row 1}$ from row 4

$$\begin{array}{c} A \qquad \qquad \mathbf{b} \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{0} & -46 & -11 & -31 \\ -\mathbf{2} & 3 & 0 & -5 \end{array} \right] \end{array} \begin{array}{c} \left[\begin{array}{c} -2 \\ 22 \\ 14 \\ 3 \end{array} \right] \end{array}$$

$$\begin{array}{c} E_{41} \cdot \downarrow \\ \left[\begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{0} & -46 & -11 & -31 \\ \mathbf{0} & 21 & 4 & 5 \end{array} \right] \end{array} \begin{array}{c} E_{41} \cdot \downarrow \\ \left[\begin{array}{c} -2 \\ 22 \\ 14 \\ -1 \end{array} \right] \end{array}$$

$$\text{row 4 :} \quad -\mathbf{2} \quad \mathbf{3} \quad \mathbf{0} \quad -5 \quad 3$$

$$\frac{(-2) \cdot \text{row 1 :}}{\quad} \quad \frac{-2}{\quad} \quad \frac{-18}{\quad} \quad \frac{-4}{\quad} \quad \frac{-10}{\quad} \quad \frac{4}{\quad}$$

$$= \text{row 4' :} \quad \mathbf{0} \quad \mathbf{21} \quad \mathbf{4} \quad 5 \quad -1$$

$$\text{elimination matrix: } E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 2: Subtract $\frac{23}{33} \cdot \text{row 2}$ from row 3

$$\begin{array}{c} \mathbf{A} \qquad \qquad \mathbf{b} \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & -46 & -11 & -31 \\ 0 & 21 & 4 & 5 \end{array} \right] \left[\begin{array}{c} -2 \\ 22 \\ 14 \\ -1 \end{array} \right] \end{array}$$

$$\text{row 3 : } \quad 0 \quad -46 \quad -11 \quad -31 \quad 14$$

$$\begin{array}{c} E_{32} \cdot \downarrow \qquad \qquad E_{32} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 21 & 4 & 5 \end{array} \right] \left[\begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ -1 \end{array} \right] \end{array}$$

$$\begin{array}{rcl} \frac{23}{33} \cdot \text{row 2} : & 0 & -46 \quad -\frac{92}{11} \quad -\frac{805}{33} \quad \frac{46}{3} \\ = \text{row 3'} : & 0 & 0 \quad -\frac{29}{11} \quad -\frac{218}{33} \quad -\frac{4}{3} \end{array}$$

elimination matrix: $E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{23}{33} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Elimination in column 2: Subtract $(-\frac{7}{22}) \cdot \text{row 2}$ from row 4

$$\begin{array}{c} \mathbf{A} \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 21 & 4 & 5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[\begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ -1 \end{array} \right] \end{array}$$

$$\begin{array}{c} E_{42} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & \frac{2}{11} & -\frac{135}{22} \end{array} \right] \end{array} \quad \begin{array}{c} E_{42} \cdot \downarrow \\ \left[\begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ 6 \end{array} \right] \end{array}$$

$$\text{row 4 : } \quad 0 \quad 21 \quad 4 \quad 5 \quad -1$$

$$\frac{(-\frac{7}{22}) \cdot \text{row 2 :}}{\quad} \quad 0 \quad 21 \quad \frac{42}{11} \quad \frac{245}{22} \quad -7$$

$$= \text{row 4' : } \quad 0 \quad 0 \quad \frac{2}{11} \quad -\frac{135}{22} \quad 6$$

$$\text{elimination matrix: } E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{7}{22} & 0 & 1 \end{bmatrix}$$

Elimination in column 3: Subtract $(-\frac{2}{29}) \cdot \text{row 3}$ from row 4

$$\begin{array}{c} \mathbf{A} \qquad \qquad \mathbf{b} \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & \frac{2}{11} & -\frac{135}{22} \end{array} \right] \end{array}$$

$$\text{row 4 : } \quad 0 \quad 0 \quad \frac{2}{11} \quad -\frac{135}{22} \quad 6$$

$$\begin{array}{c} E_{43} \cdot \downarrow \\ \left[\begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & 0 & -\frac{1147}{174} \end{array} \right] \end{array}$$

$$\begin{array}{c} E_{43} \cdot \downarrow \\ \left[\begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ \frac{514}{87} \end{array} \right] \end{array}$$

$$\begin{array}{l} (-\frac{2}{29}) \cdot \text{row 3 :} \\ \hline = \text{row 4' :} \end{array} \quad \begin{array}{ccccccc} 0 & 0 & \frac{2}{11} & -\frac{436}{957} & -\frac{8}{87} \\ 0 & 0 & 0 & -\frac{1147}{174} & \frac{514}{87} \end{array}$$

$$\text{elimination matrix: } E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{29} & 1 \end{bmatrix}$$

Back substitution to compute x_4, x_3, x_2, x_1

Matrix vector form $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & 0 & -\frac{1147}{174} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 22 \\ -\frac{4}{3} \\ \frac{514}{87} \end{bmatrix}}_{\mathbf{b}}$$

System of linear equations

$$\begin{array}{rrcrcl} 1x_1 & +9x_2 & +2x_3 & +5x_4 & = & -2 \\ & -66x_2 & -12x_3 & -35x_4 & = & 22 \\ & & -\frac{29}{11}x_3 & -\frac{218}{33}x_4 & = & -\frac{4}{3} \\ & & & -\frac{1147}{174}x_4 & = & \frac{514}{87} \end{array}$$

eq.	before substitution					after substitution		solution	
1	$1x_1$	$+9x_2$	$+2x_3$	$+5x_4$	$= -2$	$1x_1 - \frac{2527}{1147}$	$= -2$	$x_1 =$	$\frac{233}{1147}$
2		$-66x_2$	$-12x_3$	$-35x_4$	$= 22$	$-66x_2 - \frac{1892}{1147}$	$= 22$	$x_2 =$	$-\frac{411}{1147}$
3			$-\frac{29}{11}x_3$	$-\frac{218}{33}x_4$	$= -\frac{4}{3}$	$-\frac{29}{11}x_3 + \frac{224104}{37851}$	$= -\frac{4}{3}$	$x_3 =$	$\frac{3156}{1147}$
4				$-\frac{1147}{174}x_4$	$= \frac{514}{87}$			$x_4 =$	$-\frac{1028}{1147}$

