Linear Algebra

ETH Zürich, HS 2023, 401-0131-00L

The Computer Science Lens

Fast Fibonacci Numbers by Iterative (Matrix) Squaring

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Fibonacci numbers at Zürich HB

Fibonacci numbers

The sequence

			2	$+$	3	$=$	5		13	$+$	21	$=$	34		

$$
\begin{array}{ccccccccccccc}
 & & & 2 & + & 3 & = & 5 & & 13 & + & 21 & = & 34 & & \\
\hline
0 & & 1 & & 1 & & 2 & & 3 & & 5 & & 8 & & 13 & & 21 & & 34 & & 55 & \ldots \\
f_0 & & f_1 & & f_2 & & f_3 & & f_4 & & f_5 & & f_6 & & f_7 & & f_8 & & f_9, & & f_{10} & \ldots \\
\hline
 & & & f_3 & + & f_4 & = & f_5 & & f_7 & + & f_8 & = & f_9 & &
\end{array}
$$

Every number is the sum of the previous two.

Mathematical definition:

$$
\begin{aligned}
f_0 &= 0 \\
f_1 &= 1 \\
f_n &= f_{n-1} + f_{n-2}, \quad \text{if } n \geq 2
\end{aligned}
$$

The Fibonacci number f_{100}...

...is something you always wanted to know!

Here you go (Python):

```
F = [0,1]  # list containing f_0, f_1; add f_2, f_3, ..., f_100:
for n in range(2, 101):
    f_n = F[-1] + F[-2]  # f_n = sum of the previous two numbers
    print(n, ":", f_n)   # print n and f_n
    F.append(f_n)        # add f_n to the list
```

	21 : 10946	41 : 165580141	61 : 2504730781961
2 : 1	22 : 17711	42 : 267914296	62 : 4052739537881
3 : 2	23 : 28657	43 : 433494437	63 : 6557470319842
4 : 3	24 : 46368	44 : 701408733	64 : 10610209857723
5 : 5	25 : 75025	45 : 1134903170	65 : 17167680177565
6 : 8	26 : 121393	46 : 1836311903	66 : 27777890035288
7 : 13	27 : 196418	47 : 2971215073	67 : 44945570212853
8 : 21	28 : 317811	48 : 4807526976	68 : 72723460248141
9 : 34	29 : 514229	49 : 7778742049	69 : 117669030460994
10 : 55	30 : 832040	50 : 12586269025	70 : 190392490709135
11 : 89	31 : 1346269	51 : 20365011074	71 : 308061521170129
12 : 144	32 : 2178309	52 : 32951280099	72 : 498454011879264
13 : 233	33 : 3524578	53 : 53316291173	73 : 806515533049393
14 : 377	34 : 5702887	54 : 86267571272	74 : 1304969544928657
15 : 610	35 : 9227465	55 : 139583862445	75 : 2111485077978050
16 : 987	36 : 14930352	56 : 225851433717	76 : 3416454622906707
17 : 1597	37 : 24157817	57 : 365435296162	77 : 5527939700884757
18 : 2584	38 : 39088169	58 : 591286729879	78 : 8944394323791464
19 : 4181	39 : 63245986	59 : 956722026041	79 : 14472334024676221
20 : 6765	40 : 102334155	60 : 1548008755920	80 : 23416728348467685

81 : 37889062373143906
82 : 61305790721611591
83 : 99194853094755497
84 : 160500643816367088
85 : 259695496911122585
86 : 420196140727489673
87 : 679891637638612258
88 : 1100087778366101931
89 : 1779979416004714189
90 : 2880067194370816120
91 : 4660046610375530309
92 : 7540113804746346429
93 : 12200160415121876738
94 : 19740274219868223167
95 : 31940434634990099905
96 : 51680708854858323072
97 : 83621143489848422977
98 : 135301852344706746049
99 : 218922995834555169026
100 : 354224848179261915075

$$f_{100} = 354224848179261915075.$$

Let's do it faster!

Observation [Mat10]:

$$\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{pmatrix} f_{n-2} \\ f_{n-1} \end{pmatrix}, \quad \text{if } n \geq 2$$

$$\begin{aligned} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} &= A \begin{pmatrix} f_{n-2} \\ f_{n-1} \end{pmatrix} = A \left(A \begin{pmatrix} f_{n-3} \\ f_{n-2} \end{pmatrix} \right) = \cdots = A \left(\cdots \left(A \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \right) \cdots \right) \\ &= \underbrace{A \left(\cdots \left(A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \cdots \right)}_{n-1 \ A\text{'s}} = A^{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(generalized associativity!)} \end{aligned}$$

To compute f_n, we need the matrix $A^{n-1} = \underbrace{A \cdot A \cdots A}_{n-1 \text{ times}}$.

Then f_n is found in the lower right corner: recall $\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \underbrace{A^{n-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{column 2 of } A^{n-1}}$.

Fast powers by iterative squaring ($n = 100$: need A^{99})

$$
\begin{aligned}
A^{99} &= \left(A^{49}\right)^2 \cdot A = \begin{bmatrix} 135301852344706746049 & 218922995834555169026 \\ 218922995834555169026 & \color{red}{354224848179261915075} \end{bmatrix} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \color{red}{= f_{100}} \\
&\swarrow \\
A^{49} &= \left(A^{24}\right)^2 \cdot A = \begin{bmatrix} 4807526976 & 7778742049 \\ 7778742049 & 12586269025 \end{bmatrix} \\
&\swarrow \\
A^{24} &= \left(A^{12}\right)^2 = \begin{bmatrix} 28657 & 46368 \\ 46368 & 75025 \end{bmatrix} \\
&\swarrow \\
A^{12} &= \left(A^{6}\right)^2 = \begin{bmatrix} 89 & 144 \\ 144 & 233 \end{bmatrix} \\
&\swarrow \\
A^{6} &= \left(A^{3}\right)^2 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} \\
&\swarrow \\
A^{3} &= (A)^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}
\end{aligned}
$$

References

Jiří Matoušek.
Thirty-three Miniatures - Mathematical and Algorithmic Applications of Linear Algebra.
American Mathematical Society, 2010.
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