Linear Algebra
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Vector Spaces

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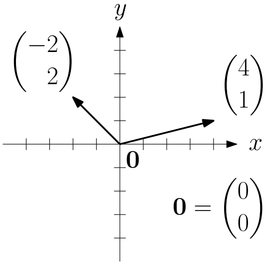
So far (Definition 1.1):

"A vector is an element of $\mathbb{R}^m$."



vectors in $\mathbb{R}^2$, drawn as arrows

The truth:

- There are also other kinds of vectors.
- "…an element of $\mathbb{R}^m$" was actually a white lie.

What “the Internet” thinks a vector is

Oxford Languages:

> *a quantity having direction as well as magnitude, especially as determining the position of one point in space relative to another.*

Chat GPT:

> *A vector is a mathematical object that has both magnitude and direction. You can think of it as a an arrow. [...]*

Wikipedia:[1].

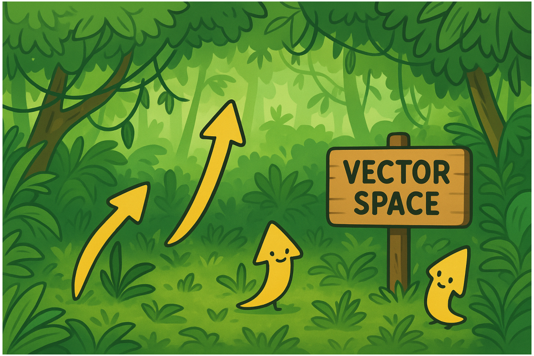
> *In mathematics and physics, vector is a term that refers to quantities that cannot be expressed by a single number (a scalar), or to elements of some vector spaces.*

[1] `https://en.wikipedia.org/wiki/Vector_(mathematics_and_physics)`, accessed on October 16, 2025

The only defininition that always works

A vector is an element of a vector space.

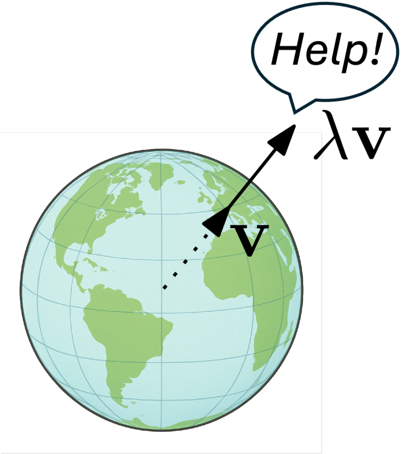
Vector space: a “natural habitat” for vectors . . .



. . . in which they can safely pursue their natural behavior:

- add up ($\mathbf{v} + \mathbf{w}$)
- scale ($\lambda \mathbf{v}$)

Not every habitat is safe for vectors. . .



Vector space, informally

A vector space is a set together with two operations: vector addition $\mathbf{v} + \mathbf{w}$ and scalar multiplication $\lambda \cdot \mathbf{v}$, each producing another element of the vector space.

These operations have to follow some rules (details will follow).

Example

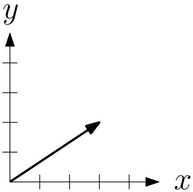
The vector space of polynomials $(x^2 + x + 1, 3x^3, 5x - 2, \ldots)$.

- $(x^2 + x + 1) \; + \; (5x - 2) \; = \; x^2 + 6x - 1$
- $5 \; \cdot \; (x^2 + x + 1) \; = \; 5x^2 + 5x + 5$

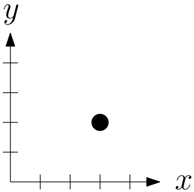
Here, the vectors are polynomials, no “magnitude” or “direction” is apparent.

The white lie: $\mathbb{R}^m$ is *not* a vector space...

$\mathbb{R}^2$ just contains "raw" pairs of numbers such as $(3, 2)$. The *meaning* can vary.



Vector



Point

Score

The truth: $(\mathbb{R}^2, +, \cdot)$ is the vector space: this is $\mathbb{R}^2$ together with the standard vector addition $(+)$ and scalar multiplication $(\cdot)$ from Definitions 1.2 and 1.3.

For that vector space, we use arrow drawings and column vector notation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Calling this vector space $\mathbb{R}^2$ is a typical and acceptable abuse of notation.

Real vector spaces, formally

A real vector space[2] is a triple $(V, +, \cdot)$ where V is a set (the vectors), and

$$\begin{array}{lcll} + & : & V \times V \to V & \text{a function (vector addition),} \\ \cdot & : & \mathbb{R} \times V \to V & \text{a function (scalar multiplication),} \end{array}$$

satisfying the following *axioms* (rules) for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all $\lambda, \mu \in \mathbb{R}$.

don't learn them by heart!

1.	$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	commutativity
2.	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	associativity
3.	There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	zero vector
4.	There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	negative vector
5.	$1 \cdot \mathbf{v} = \mathbf{v}$	identity element
6.	$(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$	compatibility of $\cdot$ and $\cdot$ in $\mathbb{R}$
7.	$\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$	distributivity over $+$
8.	$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$	distributivity over $+$ in $\mathbb{R}$

[2] "real" stands for real numbers $\lambda \in \mathbb{R}$ as scalars

Example: The vector space of polynomials

Polynomial (of degree m): formal sum of the form $\mathbf{p} = \sum_{i=0}^{m} p_i x^i, p_m \neq 0$

V	:	all polynomials	$x^2 + x + 1, 3x^3, 5x - 2, \ldots$
$+$	:	vector addition	$(x^2 + x + 1) + (5x - 2) = x^2 + 6x - 1$
$\cdot$	:	scalar multiplication	$5 \cdot (x^2 + x + 1) = 5x^2 + 5x + 5$

Vector space axioms: easy (and boring) to check...

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	
3. There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	the zero polynomial (degree $m = -1$)
4. There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	
5. $1 \cdot \mathbf{v} = \mathbf{v}$	
6. $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$	
7. $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$	
8. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$	

Yet another real vector space

Theorem 4.5: Let $\mathbb{R}^{m \times n}$ be the set of $m \times n$ matrices, with addition $A + B$ and scalar multiplication λA defined in the usual way, see Definition 2.2. Then $(\mathbb{R}^{m \times n}, +, \cdot)$ is a vector space.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

Let's prove some "obvious" facts about real vector spaces (I)

Vector space axioms:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$
4. There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $1 \cdot \mathbf{v} = \mathbf{v}$
6. $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$
7. $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
8. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

Lemma

There is only one zero vector.

Proof.

Take two zero vectors $\mathbf{0}$ and $\mathbf{0}'$. Then

$$\begin{aligned}
\mathbf{0}' &= \mathbf{0}' + \mathbf{0} && (3.\ \mathbf{0} \text{ is a zero vector}) \\
&= \mathbf{0} + \mathbf{0}' && (1.\ \text{commutativity}) \\
&= \mathbf{0} && (3.\ \mathbf{0}' \text{ is a zero vector})
\end{aligned}$$

So $\mathbf{0}$ and $\mathbf{0}'$ are equal. □

Let's prove some "obvious" facts about real vector spaces (II)

Vector space axioms:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$
4. There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $1 \cdot \mathbf{v} = \mathbf{v}$
6. $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$
7. $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
8. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

Lemma

For every vector $\mathbf{v}$, we have $0 \cdot \mathbf{v} = \mathbf{0}$.

Proof.

$$
\begin{aligned}
& 0\mathbf{v} & \\
= \; & 0\mathbf{v} + \mathbf{0} & \text{(3. zero vector)} \\
= \; & 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) & \text{(4. negative)} \\
= \; & (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) & \text{(2. associativity)} \\
= \; & (0+0)\mathbf{v} + (-0\mathbf{v}) & \text{(8. distributivity } +) \\
= \; & 0\mathbf{v} + (-0\mathbf{v}) & \text{(rules of } \mathbb{R}) \\
= \; & \mathbf{0} & \text{(4. negative)}
\end{aligned}
$$

□

Let's prove some "obvious" facts about real vector spaces (III)

Vector space axioms:

1. $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$
4. There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $1 \cdot \mathbf{v} = \mathbf{v}$
6. $(\lambda \cdot \mu)\mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$
7. $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
8. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

Lemma

Each $\mathbf{v}$ *has only one negative vector.*

Proof.

Take two negative vectors $\mathbf{u}$ and $\mathbf{u}'$ of $\mathbf{v}$. Then

$$
\begin{aligned}
\mathbf{u}' &= \mathbf{u}' + \mathbf{0} && \text{(3. zero vector)} \\
&= \mathbf{u}' + (\mathbf{v} + \mathbf{u}) && \text{(4. } \mathbf{u} \text{ is a negative)} \\
&= (\mathbf{u}' + \mathbf{v}) + \mathbf{u} && \text{(2. associativity)} \\
&= (\mathbf{v} + \mathbf{u}') + \mathbf{u} && \text{(1. commutativity)} \\
&= \mathbf{0} + \mathbf{u} && \text{(4. } \mathbf{u}' \text{ is a negative))} \\
&= \mathbf{u} + \mathbf{0} && \text{(1. commutativity)} \\
&= \mathbf{u} && \text{(3. zero vector)}
\end{aligned}
$$

So $\mathbf{u}$ and $\mathbf{u}'$ are equal. □

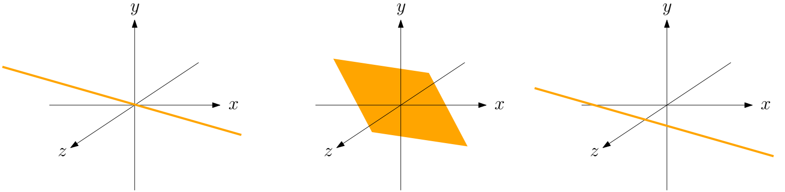
Subspaces

Definition 4.8: Let V be a vector space. A nonempty subset $U \subseteq V$ is called a *subspace* of V if the following two *axioms of a subspace* are true for all $\mathbf{v}, \mathbf{w} \in U$ and all $\lambda \in \mathbb{R}$.

(i) $\mathbf{v} + \mathbf{w} \in U$;

(ii) $\lambda \mathbf{v} \in U$.

Subspace: natural "sub-habitat" in a bigger habitat V



not a subspace!

Lemma 4.9: Let $U \subseteq V$ be a subspace of a vector space V. Then $\mathbf{0} \in U$.

Proof: Take $\mathbf{u} \in U$ (U is nonempty). By (ii), $0\mathbf{u} = \mathbf{0} \in U$.

Subspace examples: column Space, row Space, nullspace

Lemma 4.11: Let A be an $m \times n$ matrix. The column space $\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$.

Proof.

Let $\mathbf{v}, \mathbf{w}$ be in $\mathbf{C}(A)$. Then there exist vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{v} = A\mathbf{x}, \mathbf{w} = A\mathbf{y}$. Hence,

$$A(\underbrace{\mathbf{x} + \mathbf{y}}_{\in \mathbb{R}^n}) = A\mathbf{x} + A\mathbf{y} = \mathbf{v} + \mathbf{w} \quad \Rightarrow \quad \mathbf{v} + \mathbf{w} \in \mathbf{C}(A).$$

This was subspace axiom (i). For axiom (ii), let $\lambda \in \mathbb{R}$. Then

$$A(\underbrace{\lambda \mathbf{x}}_{\in \mathbb{R}^n}) = \lambda A\mathbf{x} = \lambda \mathbf{v} \quad \Rightarrow \quad \lambda \mathbf{v} \in \mathbf{C}(A).$$

Uses linearity of matrix transformations (Lemma 2.19). □

Row space $\mathbf{R}(A)$ and nullspace $\mathbf{N}(A)$ are also subspaces (Corollary 4.12, Exercise 4.13).

Subspaces are vector spaces

Lemma 4.14: Let V be a vector space, and let U be a subspace of V. Then U is also a vector space (with the same "$+$" and "$\cdot$" as V).

Proof.

The only interesting step: make sure that for all $\mathbf{u} \in U$, $-\mathbf{u} \in U$ (axiom 4).

So far we only know $-\mathbf{u} \in V$.

But $-\mathbf{u} \in U$ also holds, since $(-1)\mathbf{u} \in U$ by subspace axiom (ii)...

...and "obviously" $(-1)\mathbf{u} = -\mathbf{u}$. $\square$

More subspace examples: Polynomials

$V =$ all polynomials

- $U =$ all polynomials without a constant term (for example $3x^3 + 5x$):

$$\mathbf{p} = \sum_{i=1}^{m} p_i x^i$$

- $U =$ all quadratic polynomials (for example $5x^2 + 2x + 7$ or $2x - 6$):

$$\mathbf{p} = p_0 + p_1 x + p_2 x^2$$

  This subspace looks a lot like (is *isomorphic* to) $\mathbb{R}^3$: all (p_0, p_1, p_2).

  See Section 4.2.5 for isomorphic vector spaces.

More subspace examples: 2×2 Matrices

$V = \mathbb{R}^{2\times 2}$ (isomorphic to $\mathbb{R}^4$)

- $U =$ all symmetric matrices:

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

- $U =$ all matrices of trace 0:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } a + d = 0.$$

- $U =$ all matrices with nonnegative entries?

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } a, b, c, d \geq 0.$$

Not a subspace: $(-1)A \notin U$, violation of axiom (ii).