Exam for Randomized Algorithms Course

Tuesday, Nov 28, 9:00 - 12:00, B42 (IFW), written exam; material: lectures, exercises, reading assignments; books, notes, etc. allowed, (but no electronic devices). *It is understood that answers have to be substantiated!*

Problem 1 *Consider the following process* $\mathrm{MinAndMax}$ *applied to a finite set* A *of integers.*

$A \subseteq \mathbf{Z}$, finite.

PRECONDITION:
$A \neq \emptyset$.

POSTCONDITION:
$x = \min A$, $y = \max A$.

function MinAndMax(A)
if $\#A \leq 2$ **then**
  $(x, y) \leftarrow (\min A, \max A)$;
else
  $a \leftarrow_{\text{random}} A$;
  $(x, y) \leftarrow$ MinAndMax$(A \setminus \{a\})$;
  if $a \notin \{x..y\}$ **then**
    $(x, y) \leftarrow$ MinAndMax(A);
return (x, y);

$n \in \mathbf{N}$. *What is the expected number of operations '*$a \notin \{x..y\}$*' implied by a call with a set* A *of cardinality* n*? What is the expected number of assignments '*$(x, y) \leftarrow (\min A, \max A)$;*'?*

Problem 2 S *a finite set. Given a subset* A *of* S*, consider the following process.*

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forever do
  $a \leftarrow_{\text{random}} S$;
  $x \leftarrow_{\text{random}} \{0, 1\}$;
  if $x = 1$ **then**
    $A \leftarrow A \cup \{a\}$;
  else
    $A \leftarrow A \setminus \{a\}$;

Employ coupling to argue about how long it takes until A *is a random subset of* S*? ('Random' means 'uniformly at random from* 2^S*.')*

Problem 3 $\ell, n \in \mathbf{N}$, $\ell \geq 2$. *You are given an* ℓ*-uniform hypergraph*[1] *with vertex set* $\{1..n\}$ *and with* $m \leq 2^{\ell-2}$ *hyperedges. Show that it is possible to 2-color the vertices such that no hyperedge is monochromatic. Describe and analyze a randomized algorithm for finding such a 2-coloring in a hypergraph as described above with polynomial (in terms of* n *and* m*) expected running time.*

[1] Remember that in an ℓ-uniform hypergraph every hyperedge is an ℓ-element subset of the vertex set.

Problem 4 *Give a lower and upper bound for – ideally determine – the VC-dimension of axis-parallel closed rectangles in the plane.*

Problem 5 $d \in \mathbf{N}$. *The hypercube C_d is the graph with vertex set $\{0,1\}^d$ where two vertices (i.e. $\{0,1\}$-sequences) x and y are connected by an edge iff they differ in exactly one position. Now start a random walk at a vertex of C_d according to the following rules:*

At each step, stay at the current vertex with probability $\frac{1}{2}$;
for each neighbor u of the current vertex, move to u with probability $\frac{1}{2d}$.

How long does it take until the current vertex is a random vertex of C_d?

Problem 6 $k \in \mathbf{N}$. *An $(=k)$-SAT formula is the conjunction of clauses, each clause the disjunction of* exactly k *literals. Find some function $f : \mathbf{N} \to \mathbf{N}$ (as large as you can), such that every $(=k)$-SAT formula with at most $f(k)$ clauses is satisfiable.*

Problem 7 *You are about to buy a lottery ticket, when the vendor makes the the following offer:* "I have a set T of 500 lottery tickets, each ticket t has a distinct number $n_t \in \mathbf{Z}$. For 50 sfr you can look at them in random order. After seeing a ticket you can either take it and stop, or you throw it away (and will not be able to retrieve it later). After you decide on a ticket $\bar{t}$, then the number of Swiss franks you get is the rank of $n_{\bar{t}}$ in $\{n_t \mid t \in T\}$ (i.e. $\#\{t \in T \mid n_t \leq n_{\bar{t}}\}$). *Describe how you would proceed in order to gain as much as possible, and analyze your proposal.*