Exam for Randomized Algorithms Course

Thursday, Nov 29, 2001, 9:00 - 11:45, IFW B42 and TU Berlin MA 6-2, written exam; material: lectures, exercises, reading assignments; books, notes, etc. allowed, (but no electronic devices). *It is understood that answers have to be substantiated!* That is, you have to provide proofs for the given answers.

Problem 1 is perhaps an easy starter. For the rest I prefer not to provide a ranking, since there are no dramatic differences in difficulty.

Problem 1 *Let* $\pi = (a_1, a_2, \ldots, a_n)$ *be a permutation of* $\{1..n\}$*, uniformly at random from* $\mathcal{S}_n$*. For* $i \in \{1..n\}$ *we define random variables*

$$X_i := [a_i < \min\{a_1, a_2, \ldots, a_{i-1}\}] \quad \textit{and} \quad Y_i := [a_i < \min\{a_{i+1}, a_{i+2}, \ldots, a_n\}]$$

For which values of i*, the random variables* X_i *and* Y_i *are independent?*

Problem 2 *For a finite set* Q *of at least two points in* $\mathbf{R}^2$*, let* $\mathrm{box}Q$ *be the axis-parallel closed rectangle of smallest area containing* Q*; if* Q *is contained in a vertical or horizontal line,* $\mathrm{box}Q$ *is the shortest axis-parallel segment containing* Q.

Now, for $P \subseteq \mathbf{R}^2$ *with* $n := \#P \geq 2$ *and* $r \in \mathbf{N}$ *with* $2 \leq r \leq n$*, we consider the expectation*

$$E(\#(P \setminus \mathrm{box}R)), \quad R \textit{ uniformly at random from } \binom{P}{r}.$$

Derive upper and lower bounds for this quantity in dependence of n *and* r.

Problem 3 *Consider the following process* TopTwo *applied to a finite set* A *of integers.*

$A \subseteq \mathbf{Z}$, finite.

PRECONDITION:
$\#A \geq 2$.

function TopTwo(A)
if $\#A = 2$ **then**
  $(x, y) \leftarrow (\min A, \max A)$;
else
  $a \leftarrow_{\text{random}} A$;
  $(x, y) \leftarrow \mathrm{TopTwo}(A \setminus \{a\})$;
  if $a > x$ **then**
    $(x, y) \leftarrow \mathrm{TopTwo}(A \setminus \{x\})$;
return (x, y);

$n \in \mathbf{N}$, $n \geq 2$. *What is the expected number of operations '*$a > x$*' for an input of cardinality* n*? What is the expected number of assignments '*$(x, y) \leftarrow (\min A, \max A)$;*'?*

Problem 4 *If* $\pi = (a_1, a_2, \dots, a_n)$ *is a permutation of* $\{1..n\}$*, uniformly at random from* $\mathcal{S}_n$*, then*

$$\Pr(a_i = j \textit{ and } a_{i+1} = j+1) = \frac{1}{n(n-1)} \quad \textit{for all } i, j \in \{1..n-1\}.$$

Provide a small sample space $\Omega \subseteq \mathcal{S}_n$ *that guarantees this property under uniform sampling from* Ω.

Problem 5 *Let* $G = (V, E)$ *be graph with* t *triangles. Suppose each vertex gets assigned a number uniformly at random in* $\{1, 2, 3\}$ *– mutually independent for all vertices.*

Prove that the expected number of triangles that get three distinct numbers assigned is $\frac{2t}{9}$*. Then show how this construction can be efficiently derandomized. That is, derive a polynomial time procedure that produces an assignment with at least* $\frac{2t}{9}$ *triangles with three distinct numbers.*

Problem 6 *Consider the range space* $(\mathbf{R}^2, \mathcal{Q})$*, where* $\mathcal{Q}$ *is the set of all closed quadrilaterals in the plane, not necessarily convex. Show that this range space has finite VC-dimension. Exhibit some explicit upper for the VC-dimension.*

Problem 7 *Given a finite set* $Q \subseteq \mathbf{R}^2$*, let* αQ *be the smallest radius of a ball that contains all but one point in* Q*. To be precise,*

$$\alpha Q := \begin{cases} \inf\{r_b \mid b \textit{ is a closed ball with } \#(b \cap Q) \geq \#Q - 1\}, & \textit{if } \#Q \geq 2, \\ -\infty, & \textit{if } \#Q = 1, \textit{ and} \\ \mho, & \textit{if } Q = \emptyset, \end{cases}$$

with the ordering $\mho < -\infty$ *(and, as usual,* $-\infty < x$ *for any real number* x*). Is* (P, α) *an LP-type problem for all finite* $P \subseteq \mathbf{R}^2$*?*