Satisfiability of Boolean Formulas - Combinatorics and Algorithms
Autumn 2008

Special Exercises

Regulations:

- There will be a total of four special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX.
- You are welcome to discuss the tasks with your colleagues, but we expect each of you to hand in your own, individual writeup.
- Your solutions will be graded. The three highest out of your four achieved grades will account for 10% of your final grade for the course each (so 30% of the grade in total).
- **Note:** The last set of special assignments (Set 4) will be handed out on November 21 and has to be solved by December 5

Special Exercise Set 3

Due date: Friday, November 21, 2008 (at the beginning of the 10:15 lecture)

Reading Assignment Algorithmic Lovász Local Lemma

We hand out the paper by Robin Moser, 'A constructive proof of the Lovász Local Lemma'. You may also obtain it digitally at `http://arxiv.org/abs/0810.4812`. It is an improvement over the results presented in Chapter 2*. Your task is to read and digest the proof. Please ask as many questions as necessary to make sure that you have understood everything in depth. The material contained will be part of both the final exam and the last set of special assignments.

Problem 1 Minimal Unsatisfiable 2-CNF formulas

A CNF formula F is called *minimal unsatisfiable* if it is unsatisfiable but for every $C \in F$, the formula $F \setminus \{C\}$ is satisfiable. For the tasks below, let F be any 2-CNF that is minimal unsatisfiable.

a. Show that $\square \in \mathrm{uc}(F^{[x \mapsto 0]})$ and $\square \in \mathrm{uc}(F^{[x \mapsto 1]})$ for all $x \in \mathrm{vbl}(F)$.

b. Show that $|F| \leq 2 \cdot |\mathrm{vbl}(F)|$.

Problem 2 Properties of the Hamming cube

Let $n \in \mathbb{N}$ and $V := \{x_1, x_2, \ldots, x_n\}$ be a set of n variables. Consider the V-cube. If ϕ is any face of the V-cube, then we can associate with ϕ a clause C_ϕ such that ϕ covers exactly those vertices that represent assignments violating C_ϕ.

a. Show that for any $1 \leq i \leq 2^n$, there is a partition of the V-cube into exactly i disjoint faces. Describe what such a partition looks like.

b. Show that for any $1 \leq i \leq 2^n$, there is a CNF formula F over V consisting of exactly i clauses which is minimal unsatisfiable (cf. Problem 1 for the definition).

c. Let $\phi_1, \phi_2, \ldots, \phi_k$ be any set of faces of the V-cube. Show that if $\phi_i \cap \phi_j \neq \emptyset$ for all $i \neq j$, then $\bigcap_{i=1}^{n} \phi_i$ is a non-empty face of the cube.

d. Determine the minimum number of clauses needed to build a CNF formula over the variable set (exactly) V which is satisfied if and only if exactly t variables are set to true, for any given $1 \leq t \leq n$.

Problem 3 Consecutive bits

Let us say that a 3-CNF formula F over the variable set $\{x_1, x_2, \dots, x_n\}$ is *consecutive*, if for each clause $C \in F$, there is an index $i_C \in [n]$ such that $\mathrm{vbl}(C) = \{x_{i_C}, x_{i_C+1}, x_{i_C+2}\}$. Note that multiple clauses can have the same variable set.

Let a *consecutive 3-falsifier* be defined just like a 3-falsifier, with the additional restriction that the sets of positions it inspects in the certificate are consecutive, i.e. $\{i, i+1, i+2\}$ for some i. Let $\mathcal{FP}_{3c}$ be the class of languages for which there is a polynomial consecutive 3-falsifier.

a. Prove that there is an algorithm that decides satisfiability of a consecutive 3-CNF formula in polynomial time.

b. Prove that $\mathcal{FP}_{3c} = \mathcal{P}$.