Exam "Satisfiability of Boolean Formulas"

February 21, 2007, 9:00am – 11:00am, IFW A 36

Each of the six problems is granted the same number of points. You don't need to solve all of them: Completely solving five out of these problems will already give you the best grade. Don't forget to put your full name on the first page, and your initials on each of the other pages. You may use English and German, whatever you prefer.

Problem 1 — Consider the formula $F = \{\{\bar{x}\}, \{y, z\}, \{y, \bar{z}\}, \{x, \bar{y}, z\}, \{x, \bar{y}, \bar{z}\}\}$.

1.1 Give a resolution proof that F is unsatisfiable.
1.2 Compute $\sum_{C \in F} 2^{-|C|}$.
1.3 Give an assignment under which at least 2 clauses of F are not satisfied.
1.4 Pick each variable's value uniformly at random from $\{0, 1\}$, independently of each other. What is the expected number of satisfied clauses?

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Problem 2

2.1 Let x be some distinguished variable. An All-x-3-CNF formula F is a 3-CNF formula over some variable set V with $x \in V$ such that each clause of F contains either x or $\bar{x}$ (besides two other literals, of course). Prove that it can be decided in polynomial time whether an All-x-3-CNF formula is satisfiable.
2.2 Generalize the former problem. Fix some set of variables U of constant size. An All-U-3-CNF formula F is a 3-CNF formula over some variable set V with $U \subseteq V$ such that each clause of F contains at least one literal from $U \cup \bar{U}$ (where $\bar{U} = \{\bar{u} \mid u \in U\}$). Prove that it can be decided in polynomial time whether an All-U-3-CNF formula is satisfiable. What is the running time of your algorithm? How does it depend on $|U|$?

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Problem 3 — Let $S \subseteq \{0, 1\}^n$ be nonempty. Show that there exists a partition of the n-dimensional cube $\{0, 1\}^n$ into faces $\varphi_1, \varphi_2, \ldots, \varphi_{|S|}$ such that each φ_i contains exactly 1 element of S.

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Problem 4 — A *monotone 2,3-CNF formula* is a formula consisting of 2-clauses and 3-clauses, where each 3-clause contains only positive literals, e.g. $\{x, y, z\}$, and each 2-clause contains only negative literals, e.g. $\{\bar{x}, \bar{y}\}$.

4.1 Give an example of an unsatisfiable monotone 2,3-CNF formula, and explain why it is unsatisfiable.
4.2 Show that in every monotone 2-3-CNF one can satisfy at least a fraction of $1-p_0^2 \approx 0.815$ of all clauses, where $p_0 \approx 0.43$ is the unique real root of p^3-2p^2+3p-1.
4.3 Let *Monotone 2,3-SAT* be the problem of deciding whether a given monotone 2,3-CNF is satisfiable. Show that Monotone 2,3-SAT is NP-complete (Hint: Consider Graph 3-Colorability, which is NP-complete).

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Problem 5 — Sample a random 1-CNF formula $F_{n,p}$ over n variables as follows: Choose each of the $2n$ possible clauses $\{x_1\}, \ldots \{x_n\}, \{\bar{x}_1\}, \ldots, \{\bar{x}_n\}$ with probability p, independently of each other (this sampling is different from the sampling we did in the lecture).

5.1 Give a simple (necessary and sufficient) criterion for a 1-CNF formula to be satisfiable
5.2 How many satisfiable 1-CNF formulas over n variables are there?
5.3 What is the probability that $F_{n,p}$ is satisfiable?
5.4 What is the expected number of satisfying assignments of $F_{n,p}$?

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Problem 6 — Recall the following fact: A k-CNF formula where every variable occurs in at most k clauses is satisfiable. The proof is a direct application of Hall's Theorem. Prove the more general statement:

Claim: Let F be a CNF formula over variable set V. For each variable v, let

$$\text{weight}(v) := \sum_{C \in F} \frac{1}{|C|} .$$

If $\text{weight}(v) \leq 1$ for all variables in V, then F is satisfiable.

Hint: Consider the bipartite graph on V and F where $(v, C) \in V \times F$ is an edge iff $v \in C$ or $\bar{v} \in C$. Now define a weight function on the edges of this graph. If you do it right, the statement follows again from Hall's Theorem.

Hall's Theorem: Let $G = (A, B, E)$ be a bipartite graph. If for any subset $A' \subseteq A$, the set of neighbors of A', i.e. $\{b \in B \mid \exists a \in A' : \ (a, b) \in E\}$, has at least as many elements as A', then there exists a matching in G that matches every vertex in A.

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