

Institute of Theoretical Computer Science
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June 1, 2012

Satisfiability of Boolean Formulas

Spring 2012
Final Exam

Candidate:

First name:

Last name:

Student ID (Legi) Nr.:

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature:

General remarks and instructions:

1. You can solve the 4 exercises in any order. Not all points are necessary in order to get the best grade. Usually, it pays off to solve fewer tasks cleanly rather than all tasks. Select wisely, read all tasks carefully first. They are not ordered by difficulty or in any other meaningful way.
2. Check your exam documents for completeness (2 cover pages and 2 pages containing 4 exercises).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. The exam is open-book. All written material is allowed but no electronics of whatever sort.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. No points will be awarded for unfounded or incomprehensible solutions (except in the multiple-choice parts). You can write your solution in English or German.**
9. You do not need to reprove (but can reference) things that were already proved in the lecture or in any of the exercises treated in class. But if you want to prove something *different* then you must point out all details that need to be done differently in your proof.
10. Make sure to write your student-ID (**Legi-number**) on **all** the sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(30)	
2	(30)	
3	(30)	
4	(30)	
Σ	(120)	

Exercise 1 - Multiple Choice (30 Pts)

Consider the following 5 claims and mark the corresponding boxes. Grading: 3 points for a correct marking without a correct justification, 6 points for a correct marking with a correct short justification, and -3 points for a wrongly marked box (you will receive non-negative total points in any case).

Note: You may write your justification on an additional sheet if you need more space. But do not spend too much time on lengthy arguments; keywords and rough sketches are sufficient here.

- (a) Suppose that in a given class $\mathcal{L}$ of (≤ 3) -CNF formulas, each formula $F \in \mathcal{L}$ over n variables admits a set $V_F \subseteq \text{vbl}(F)$ of variables of size $|V_F| = \mathcal{O}(\log n)$ such that each clause $C \in F$ has some variable from V_F . Then satisfiability of any formula in $\mathcal{L}$ can be decided in polynomial time even if $P \neq NP$.

☐ True ☐ False

Justification:
.....

- (b) Let F be a $(\leq k)$ -CNF formula (k constant) on n variables V which admits a unique satisfying assignment. Consider **ppz** running on this formula. If over uniform choice of a permutation π of V , the average number of variables that need to be guessed is $\Theta(n)$, then **ppz** has exponentially small success probability on F .

☐ True ☐ False

Justification:
.....

- (c) The *majority* function over $n \geq 7$ bits $x_1, x_2, \dots, x_n$ is true if more than half of the input bits are true. This function can be expressed as an (≤ 3) -CNF over $x_1, x_2, \dots, x_n$ without adding auxiliary variables.

☐ False ☐ True

Justification:
.....

- (d) There exists a code $\mathcal{C} \subseteq \{0, 1\}^V$ of size polynomial in n of assignments over variables $V = \{x_1, x_2, \dots, x_n\}$ with the property that for all possible $(\leq \log n)$ -clauses D , some assignment $\alpha \in \mathcal{C}$ violates D .

☐ False ☐ True

Justification:
.....

- (e) Let F be an (≤ 3) -CNF formula over V and $\alpha \in \text{sat}_V(F)$ and $x \in V$ some variable. For the probability of completing a partial assignment α_0 to α given as

$$p(\alpha_0) := \Pr[\text{ppsz}(F, V, \alpha_0, |F|) \text{ returns } \alpha]$$

we have that $p(\{x \mapsto \alpha(x)\}) = p(\emptyset)$ if and only if x is frozen in F .

☐ True ☐ False

Justification:
.....

Exercise 2 - Partial Satisfaction (30 Pts)

Let us call a weighted CNF formula (F, μ) on variables $V := \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ *allowed* if it contains only

- k -clauses over the variables $x_1, x_2, \dots, x_n$
 - k -clauses over the variables $y_1, y_2, \dots, y_n$
 - 2-clauses of the form $\{\bar{x}_i, y_i\}$ or $\{x_i, \bar{y}_i\}$ where $1 \leq i \leq n$
- (a) Prove that allowed formulas always admit an assignment satisfying at least $1 - 1/2^k$ of the weight and that this is tight, i.e. that there exist allowed formulas where no more than $1 - 1/2^k$ of the weight can be simultaneously satisfied.
- (b) Describe a polynomial-time algorithm which, on input (F, μ) produces an assignment satisfying at least $1 - 1/2^k$ of the weight.
- (c) Suppose that (F, μ) additionally has the property that every variable occurs at most $2^k/(8k)$ times. Prove that under this additional hypothesis, F is satisfiable, and describe a polynomial time algorithm producing a satisfying assignment under this guarantee.

Exercise 3 - Worst Case Formulas for Schöning (30 Pts)

Consider the chain-like 3-CNF formula $F_n^\sim$ over variables $\{x_1, x_2, \dots, x_n\}$ which you have seen multiple times: it consists of all clauses over variables $\{x_1, x_2, x_3\}$ containing at least one positive literal and additionally the clauses $\{\bar{x}_{i-2}, \bar{x}_{i-1}, x_i\}$ for $4 \leq i \leq n$. Prove that the success probability of a single run of Schöning's algorithm (doing the usual $3n$ steps) on $F_n^\sim$ is always $(\frac{3}{4})^{n+o(n)}$, irrespective of what strategy is used for selecting the clauses to be fixed in each iteration.

HINT: Use the “cuplrit method” and consider each variable which needs to be fixed as a separate process which must conclude well-before the algorithm surrenders. What is the probability that one of these processes concludes at all in finite time? Think carefully about how precise your bounds have to be, you can spare yourself a lot of work by making generous estimates.

Exercise 4 - Randomized Satisfiability Encoding (30 Pts)

Let F be an (≤ 3) -CNF formula over n variables V which has the unique satisfying assignment α^* . Suppose further that $U \in \binom{V}{n/2}$ is a subset of half the variables selected uniformly at random from all such sets. Consider the (random) partial assignment $\alpha_U := \alpha^*|_U$ which sets all the variables from U as in α^* and leaves all remaining variables untouched. Prove that the **pps** algorithm (where we use $D = \log n$ as usual) has a subexponential probability of success when given $F^{[\alpha_U]}$ as input. Note that the randomness in this probability is over both the choice of U and the randomness used by **pps**.