Exam “Satisfiability of Boolean Formulas”

December 18, 2009, 9:00am – 11:00am, CAB G 52

Each of the six problems is granted the same number of points. You don’t need to solve all of them: Completely solving five out of these problems will already give you the best grade. Don’t forget to put your full name on the first page, and your initials on each of the other pages. You may use English and German, whatever you prefer.

Problem 1 — Ironing Your CNF Formulas

Let F be a CNF formula over the variable set V. For a set $U \subseteq V$, we denote by $\mathrm{iron}_U(F)$ the formula resulting from replacing all negative literals over U in F by their positive counterpart. For example,

$$\mathrm{iron}_{\{x,y\}}\left(\{\{x, \bar{y}, z\}, \{\bar{x}, \bar{y}, \bar{z}\}\}\right) = \{\{x, y, z\}, \{x, y, \bar{z}\}\}.$$

1.1 Show that for any CNF formula F over the variable set V and any set $U \subseteq V$, it holds that $|\mathrm{sat}_V(F)| \leq |\mathrm{sat}_V(\mathrm{iron}_U(F))|$.

1.2 Give an example of a CNF formula F over the variable set V such that $\mathrm{sat}_V(F) \not\subseteq \mathrm{sat}_V(\mathrm{iron}_U(F))$.

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Problem 2 — Partial Satisfiability with a Catch

We call a clause *happy* if it is (i) a 3-clause consisting of only positive literals (e.g. $\{x, y, z\}$), or (ii) a 3-clause consisting of only negative literals (e.g. $\{\bar{x}, \bar{y}, \bar{z}\}$), or (iii) a 2-clause consisting of one positive and one negative literal (e.g. $\{x, \bar{y}\}$). A CNF formula is happy if all its clauses are happy.

2.1 Show that for every happy formula F with clause weights $w : F \to \mathbb{R}_0^+$ there is an assignment α satisfying at least a $(1 - \phi^3)$-fraction of its weight, where $\phi \approx 0.619$ is the unique solution in $[0, 1]$ of the equation $\phi^2 + \phi - 1 = 0$ (by the way, $1 - \phi^3 \approx 0.764$).

2.2 Show that this bound is *not* tight: That is, show that there exists a $\gamma > \phi$ such that every happy formula with clause weights has a truth assignment satisfying a γ-fraction of its weight.

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Problem 3 — Joint Bicolorability

We define a decision problem called Joint Bicolorability: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ over the same vertex set, is there a coloring $c : V \to \{1, 2\}$

such that G_1 contains no edge that is monochromatically 1, and G_2 contains no edge that is monochromatically 2? Formally, for $i \in \{1, 2\}$ and $u, v \in V$, $c(u) = c(v) = i$ implies $\{u, v\} \notin E_i$. If such a coloring exists, we say G_1 and G_2 are *jointly bicolorable.*

3.1 Show that if every vertex has degree at most 1 in G_1 and degree at most 1 in G_2, then G_1 and G_2 are jointly bicolorable.

3.2 Show that JOINT BICOLORABILITY is in P.

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Problem 4 — Max-2-SAT

4.1 Consider the following (≤ 2)-CNF formula:

$$G_{x,y,z} := \{\{x\}, \{y\}, \{z\}, \{u\}, \{\bar{x}, \bar{y}\}, \{\bar{y}, \bar{z}\}, \{\bar{x}, \bar{z}\}, \{x, \bar{u}\}, \{y, \bar{u}\}, \{z, \bar{u}\}\}.$$

Prove the following statement: If we set x, y, z all to 0, then we can satisfy at most 6 of the 10 clauses of $G_{x,y,z}$. Otherwise, if at least one variable of $\{x, y, z\}$ is set to 1, we can set u in such a way that 7 of 10 clauses are satisfied, and we can never satisfy more than 7 clauses.

4.2 Max-2-SAT is the following decision problem: Given an (≤ 2)-CNF formula F with integer clause weights $\mu : F \to \mathbb{N}_0$ and an integer k, does there exist an assignment α such that $\mu^{[\alpha]}(F) \geq k$?

Using the formula $G_{x,y,z}$ from above, show that MAX-2-SAT is NP-hard. To do this, show how to transform a 3-CNF formula F into a triple (F', μ, k) where F' is an (≤ 2)-CNF formula, $\mu : F' \to \mathbb{N}_0$ is a weight function, and k is an integer, such that there exists an assignment α with $\mu^{[\alpha]}(F) \geq k$ if and only if F is satisfiable.

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Problem 5 — Perfect Implementation

Recall the definition from Special Assignment Set 3: Let $f : \{0, 1\}^V \to \{0, 1\}$ be a boolean function over the variable set V. Let F be a CNF formula over variable set V' with $V' \supseteq V$. We say F *perfectly implements* f if

$$\forall \alpha : V \mapsto \{0, 1\} : \quad f(\alpha) = 1 \iff F^{[\alpha]} \text{ is satisfiable.}$$

A CNF formula F is called *Horn* if every clause in F contains at most one positive literal.

5.1 Let F be a Horn formula over variable set V. Show that for any $\alpha, \beta \in \mathrm{sat}_V(F)$, it holds that $\alpha \wedge \beta \in \mathrm{sat}_V(F)$ as well.

5.2 Show that $x \vee y$ cannot be perfectly implemented by a Horn formula.

5.3 Which of the following four functions can be perfectly implemented by a Horn formula: (i) $x \vee y \vee z$, (ii) $x \vee y \vee \bar{z}$, (iii) $x \vee \bar{y} \vee \bar{z}$, (iv) $\bar{x} \vee \bar{y} \vee \bar{z}$? Justify your answers.

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Problem 6 — Resolution Free Formulas

Recall the concepts of single conflicts from the second special assignment set: Two clauses C, D have a *single conflict* if $|C \cap \bar{D}| = 1$, where $\bar{D} := \{\bar{u} \mid u \in D\}$. We say a CNF formula F is *resolution free* if no two clauses of F have a single conflict. Furthermore, a CNF formula is *subsumption free* if there are no clauses $C, D \in F$ with $C \subsetneq D$.

6.1 Prove the following statement: If α is a partial assignment, and F is resolution free, then $F^{[\alpha]}$ is resolution free, as well.

6.2 Let F be resolution free and set $k := \min\{|C| \mid C \in F\}$. Show that if α is a partial assignment defined on at most $k - 1$ variables, then $F^{[\alpha]}$ is satisfiable.

6.3 Let F and G be CNF formulas over a variable set V, and suppose that both are resolution free and subsumption free. Show that if $F \equiv G$, then $F = G$.

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