Exam "Satisfiability of Boolean Formulas"
December 22, 2010, 8:00am – 10:00am, CAB G 56

Each of the six problems is granted the same number of points. You don't need to solve all of them: Completely solving five out of these problems will already give you the best grade. Don't forget to put your full name on the first page, and your initials on each of the other pages. You may use English and German, whatever you prefer.

1 Randomize It!

Suppose $\mathcal{A}$ is an algorithm that reads an (≤ 3)-CNF F as input, runs in polynomial time and, provided that $n := |\mathrm{vbl}(F)| \geq 4$, returns a pair (F_1, F_2) of (≤ 3)-CNF formulas such that

$$F \text{ is satisfiable } \iff F_1 \text{ is satisfiable or } F_2 \text{ is satisfiable.}$$

and $|\mathrm{vbl}(F_1)| \leq n-2$ and $|\mathrm{vbl}(F_2)| \leq n-4$.
For the following two problems, write your algorithms in pseudocode.

Problem 1.1 *Design a deterministic algorithm that, using $\mathcal{A}$, solves 3-SAT in time $O(a^n \mathrm{poly}(n))$, where $a \approx 1.27202$ is the unique positive root of $a^4 - a^2 - 1$.*

Problem 1.2 *Design a randomized algorithm that uses $\mathcal{A}$, runs in polynomial and, given a satisfiable (≤ 3)-CNF formula, finds a satisfying assignment with probability at least p^n, where $p = 1/a$ for the a in Problem 1.1.*

2 Happy Formulas

A clause C is *happy* if either

(i) $|C| = 4$,

(ii) or) $|C| = 2$ and C consists of one positive and one negative literal.

Problem 2.1 *Show that for every weighted happy CNF formula F with real non-negative clause weights $\mu : F \to \mathbb{R}_0^+$ there is an assignment α satisfying at least a $(1-p^4)$-fraction of its weight, where $p \approx 0.682$ is the unique root in $[0,1]$ of the polynomial $1 - p - p^3$.*

3 Parallel Clauses

Two clauses C, D are *parallel* if $\mathrm{vbl}(C) = \mathrm{vbl}(D)$. A CNF formula F is complete if (i) F is unsatisfiable and (ii) $\sum_{C \in F} 2^{-|C|} = 1$.

Problem 3.1 *Show that for every $m \in \mathbb{N}$ there is a complete CNF formula F with $|F| = m$.*

Problem 3.2 *Show that if F is complete and α is some (partial) assignment, then $F^{[\alpha]}$ is complete, too.*

Problem 3.3 *Suppose* F *is complete,* $|F| \geq 2$*, and there is a clause* $C \in F$ *with* $|\mathrm{vbl}(C)| = |\mathrm{vbl}(F)|$*. Show that there is at least one more clause* $D \in F$ *with the same property, i.e., some other clause* D *is parallel to* C*.*

Problem 3.4 *Show that if* F *is complete and* $|F| \geq 2$*, then there are parallel clauses* $C, D \in F$*,* $C \neq D$*.*

Hint: Let $C \in F$ be a largest clause. Then obtain F' by setting all variables in $\mathrm{vbl}(F) \backslash \mathrm{vbl}(C)$ to 1 and apply Problem 3.3

4 Performing an Execution by Hand

In this problem, you are asked to perform an execution of ppz and ppsz by hand. Consider $V = \{x_1, x_2, x_3, x_4\}$ and the formula

$$F = \{\{x_1, \bar{x}_2\}, \{x_2, \bar{x}_4\}, \{x_1, x_3, \bar{x}_4\}, \{x_1, x_4\}, \{\bar{x}_1, x_4\}\} .$$

Furthermore, let $\alpha = (1, 1, 1, 1)$ and $\pi = (x_1, x_2, x_3, x_4)$.

Problem 4.1 *Execute the function* $\mathrm{encode}(\alpha, F, V, \pi)$ *from page 97 of the lecture notes. That is, write down the formula after each iteration of the loop, and show how the string* ϵ *evolves.*

Problem 4.2 *Perform* $\mathrm{ppsz}(F, \pi, \alpha, 2)$*. Show which variables are* forced *and which are* guessed*, respectively. The procedure* ppsz *is on page 2 of the handout for Chapter 6*.*

Problem 4.3 *Choose* $\beta \leftarrow_{\text{random}} \{0, 1\}^V$ *and* $\pi \leftarrow_{\text{random}} S_V$*. What is the probability that* $\mathrm{ppsz}(F, \pi, \beta, 3) = (1, 1, 1, 1)$*?*

5 Resolution Free Formulas

Two clauses C and D have a *single conflict* if $|C \cap \bar{D}| = 1$ where $\bar{D} := \{\bar{u} \mid u \in D\}$. We say a CNF formula F is *resolution free* if no two clauses of F have a single conflict. In the following, let F be a resolution free CNF over the variable set V, where $|V| = n$, $|F| = m$, and $\square \notin F$.

Problem 5.1 *Suppose we call* $\mathrm{rfb}(F, m, \alpha)$*, defined on page 111. What is the probability that this returns a satisfying assignment? Justify your answer.*

Problem 5.2 *Show that a call to* $\mathrm{sb}(F, \alpha, n)$*, defined on page 105, takes polynomial time in* n*, for any assignment* α *on* V*.*

6 Constraint Satisfaction

Recall that an (ℓ, k)-CSP is a CSP in which every variable has a color list of size ℓ and every constraint has size k. For the following problem, give a high-level description of your algorithm. That is, do not write pseudocode.

Problem 6.1 *Let* $k \in \mathbf{N}$ *be constant. Describe a deterministic algorithm running in polynomial time that reads a* $(16, k)$*-CSP* F *over* n *variables as input and outputs a* $4k$*-CNF formula* F' *over* $4n$ *variables, such that* F *is satisfiable if and only if* F' *is satisfiable.*