Institute of Theoretical Computer Science Tuesday, March 10, 2015 (Modified on March 11)
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Satisfiability of Boolean Formulas *Spring 2015*
Special Assignment Set 1

- The solution is due on **Tuesday, March 24, 2015, 13:15 (strict!)**. Please bring a print-out of your solution with you to the exercise session. If you cannot attend, you may alternatively send your solution as a PDF to `jerri.nummenpalo@inf.ethz.ch`. We will send out a confirmation that we have received your file. Make sure you receive this confirmation within the day of the due date, otherwise complain timely.
- Please solve the exercises carefully and then write a nice and complete exposition of your solution using a computer, where we strongly recommend to use LaTeX. A tutorial can be found at `http://www.cadmo.ethz.ch/education/thesis/latex`.
- For geometric drawings that can easily be integrated into LaTeX documents, we recommend the drawing editor IPE, retrievable at `http://ipe7.sourceforge.net/` in source code and as an executable for Windows.
- You are welcome to discuss the tasks with your colleagues, but we expect each of you to hand in your own, individual write-up. **You should not share your write-up or (even worse) your source code!**
- There will be three special assignments this semester. Each of them will be graded and the average grade will contribute 30% to your final grade.
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer" or "justify intuitively", then a formal proof is **always** required.
- As with all exercises, the material covered in special assignments is relevant for the final exam.

Exercise 1 (Weighed 2-satisfiable formulas) *(30 points)*

Let (F, μ) be a weighed CNF formula where F is 2-satisfiable and $\mu : F \to W$ for some set of weights $W \subseteq \mathbb{R}_0^+$. In Theorem 2.4 and in Theorem 2.6 we proved how large a proportion of the total weight $\mu(F)$ is always satisfiable when $W = \mathbb{R}_0^+$ and $W = \{1\}$ respectively. In this exercise we look at different weight sets W and try to answer the same question.

(a) Let $W = [0, 1]$, the unit interval. How much of the total weight can we guarantee to satisfy?

(b) Let $W = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}_{>0}$. Show that if we restrict the 1-clauses to have weight 1, then there exists an assignment α with $\mu^{[\alpha]}(F) \geq \frac{2}{3}\mu(F)$. Hint: Modify the proof of Theorem 2.6. You may find it useful to look at the article *Käppeli, Claudia, and Dominik Scheder. "Partial satisfaction of k-satisfiable formulas." Electronic Notes in Discrete Mathematics 29 (2007): 497-501.* and modify the proof of their Theorem 3.4. Note that their proof is the same as the proof of our Theorem 2.6, but the exposition there might be better suited for this exercise.

(c) Let $W = [1, k]$ some $k \in \mathbb{N}_{>0}$. Can you use the same proof you used for part (b)? If yes, does anything change? If not, what breaks down?

These two bonus questions are for Exercise 1. We haven't decided on a grading scheme for them, but it is possible to attain bonus points by answering them.

Bonus 1: Let $W = \{1, 2\}$. Can you find a 2-satisfiable F for which no assignment satisfies at least $\frac{2}{3}$ of the total weight or prove that no such formula exists?
Bonus 2: Intuitively, letting $W = [1, k]$ and increasing k from 1 to infinity we would expect to have

a result which interpolates between Theorems 2.4 and 2.6. You can try and prove such a result. The second bonus question is: Try to come up with some possible research topic or research question related to partial satisfaction of CNF formulas.

Exercise 2 (NAE-Satisfiability) *(20 points)*

A CNF formula F is said to be *Not-All-Equal satisfiable*, or *NAE-satisfiable* for short, if there exists an assignment for it such that in every clause at least one literal evaluates to true and at least one literal evaluates to false.

(a) Give a 2-CNF formula with 2 clauses that is not NAE-satisfiable (and demonstrate that it really isn't!).

(b) Give a 3-CNF formula with 4 clauses that is not NAE-satisfiable (and demonstrate that it really isn't!).

(c) Show that every k-CNF formula with less than 2^{k-1} clauses is NAE-satisfiable.

(d) Show that for every k, there exists a k-CNF formula with 2^{k-1} clauses which is not NAE-satisfiable.

Exercise 3 (Only double conflicts) *(20 points)*

Let F be a CNF formula with $\Box \notin F$ such that every pair of clauses $C, D \in F$ either have no complementary literals, or they have at least two pairs of complementary literals, but never exactly one. For $C \in F$ define $\overline{C} = \{\overline{u} \mid u \in C\}$. The condition on F then states that for all $C, D \in F$ we have that

$$|C \cap \overline{D}| \neq 1.$$

(a) Prove that F is satisfiable.

(b) Exhibit a polynomial-time algorithm that finds a satisfying assignment for such a formula.