Institute of Theoretical Computer Science Tuesday, April 14, 2014
Emo Welzl, lecturer, and Jerri Nummenpalo, teaching assistant

Satisfiability of Boolean Formulas *Spring 2015*
Special Assignment Set 2

- The solution is due on **Tuesday, April 28, 2015, 13:15 (strict!)**. Please send your solution as a PDF to `njerri@inf.ethz.ch`. We will send out a confirmation that we have received your file. Make sure you receive this confirmation within the day of the due date, otherwise complain timely. Alternatively, you can return your solution in the exercise session.
- Please solve the exercises carefully and then write a nice and complete exposition of your solution using a computer, where we strongly recommend to use LaTeX. A tutorial can be found at `http://www.cadmo.ethz.ch/education/thesis/latex`.
- For geometric drawings that can easily be integrated into LaTeX documents, we recommend the drawing editor IPE, retrievable at `http://ipe7.sourceforge.net/` in source code and as an executable for Windows.
- You are welcome to discuss the tasks with your colleagues, but we expect each of you to hand in your own, individual write-up. **You should not share your write-up or (even worse) your source code!**
- There will be three special assignments this semester. Each of them will be graded and the average grade will contribute 30% to your final grade.
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer" or "justify intuitively", then a formal proof is **always** required.
- As with all exercises, the material covered in special assignments is relevant for the final exam.

Exercise 1 (Monotone CNF Complexity) *(50 points)*

Consider the set $\{0,1\}^n$ equipped with the natural partial order: For $\alpha, \beta \in \{0,1\}^n$ we write $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $1 \le i \le n$. We will write $\alpha < \beta$ if $\alpha \le \beta$ and $\alpha \ne \beta$. A boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is called *monotone* if

$$\forall\, \alpha, \beta \in \{0,1\}^n : \alpha \le \beta \Rightarrow f(\alpha) \le f(\beta)$$

A CNF formula F is *monotone* if every clause contains only positive literals. For a boolean function f, define the *CNF complexity* $c(f)$ as

$$c(f) = \min\{|F| \mid \text{F is a CNF formula, and } F \equiv f\}\,.$$

Similarly, for a monotone boolean function f define the *monotone CNF complexity* $mc(f)$ as

$$mc(f) = \min\{|F| \mid \text{F is a monotone CNF formula, and } F \equiv f\}\,.$$

Our goal is to show that for a monotone boolean function f we have that $c(f) = mc(f)$. Towards that end call an assignment $\alpha \in \{0,1\}^n$ *maximally unsatisfying* for a boolean function f if $f(\alpha) = 0$ and $f(\beta) = 1$ for every $\beta > \alpha$. Define

$$\textsc{MaxUnsat}(f) = \{\alpha \in \{0,1\}^n \mid \alpha \text{ is maximally unsatisfying for } f\}\,.$$

a) Show that $c(f) = |\textsc{MaxUnsat}(f)|$ for any monotone boolean function f.

b) Let f be a monotone boolean function. Show that $c(f) = mc(f)$.

c) Reconsider Exercise 1.10 on page 10 in the lecture notes. For $n \in \mathbb{N}$ let

$$D = (x_{11} \wedge x_{12}) \vee (x_{21} \wedge x_{22}) \vee \ldots \vee (x_{n1} \wedge x_{n2})$$

be a DNF formula on $2n$ variables. Give a short but formal proof that every CNF formula F equivalent to D has at least 2^n clauses and provide a CNF formula that matches the bound.

d) Which monotone boolean function maximizes $c(f)$? You can try and prove the result on your own, but you can also give a reference to a proof. It's not too hard to get intuition on what the right answer should be. There is a not too long proof but it uses certain nontrivial results from combinatorics, so it is most likely easiest to find some reference. To do that you need to translate the problem into a different language. Hint: boolean lattices.

Exercise 2 (Neighborhood graph) *(50 points)*

Let F be a CNF formula with m clauses over a variable set V with $|V| = n$. The *simple neighborhood graph* of F is a simple graph with the clauses of F as vertices and edges between clauses if they share a complementary literal. Formally, the simple neighborhood graph of F is the ordered pair (F, E) where $E = \{(C, D) \in F \times F \mid \exists u \in V \cup \overline{V} : u \in C, \overline{u} \in D\}$.

The *neighborhood graph with multiplicities* of F is an ordered triplet (F, E, w) where E is as above and $w : E \to \mathbb{N}$ is defined for every $(C, D) \in E$ by $w(C, D) = |\{u \in V \cup \overline{V} \mid u \in C, \overline{u} \in D\}|$ and it is called the *multiplicity* of E. That is, for every edge we also count the number of occurrences of the complementary literals between the two clauses. You can also think of the neighborhood graph with multiplicities as a multigraph (with no loops).

a) Show how the simple neighborhood graph determines the minimum number of clauses in a CNF formula that are satisfied by a total assignment.

b) What can you say about a CNF formula, where all edges have even multiplicity in the neighborhood graph with multiplicities?

c) Show that every multigraph is the neighborhood graph with multiplicities of some CNF formula.

d) Show that the neighborhood graph with multiplicities can be decomposed into n complete bipartite graphs. Define also in what sense it decomposes.

e) Show that the simple neighborhood graph has chromatic number at most 2^n, i.e. it has a proper coloring with 2^n colors. What can you say about the chromatic number if the underlying formula is a k-CNF formula?

f) Let the simple neighborhood graph of F be a complete graph. Prove or disprove: F is unsatisfiable if and only if the neighborhood graph with multiplicities of F equals the simple neighborhood graph of F i.e. the multigraph has no multiple edges.

g) If the simple neighborhood graph of an unsatisfiable CNF formula F is the complete graph, then what can you say about the sum $\sum_{C \in F} 2^{-|C|}$?

Note: One could also formulate the Lovász Local Lemma in terms of the neighborhood graph...