Institute of Theoretical Computer Science
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Tuesday, May 5, 2015

Satisfiability of Boolean Formulas *Spring 2015*
Special Assignment Set 3

- The solution is due on **Tuesday, May 19, 2015, 13:15 (strict!)**. Please send your solution as a PDF to `jerri.nummenpalo@inf.ethz.ch`. We will send out a confirmation that we have received your file. Make sure you receive this confirmation within the day of the due date, otherwise complain timely. Alternatively, you can return a printed copy of your solutions.
- Please solve the exercises carefully and then write a nice and complete exposition of your solution using a computer, where we strongly recommend to use LaTeX. A tutorial can be found at `http://www.cadmo.ethz.ch/education/thesis/latex`.
- For geometric drawings that can easily be integrated into LaTeX documents, we recommend the drawing editor IPE, retrievable at `http://ipe7.sourceforge.net/` in source code and as an executable for Windows.
- You are welcome to discuss the tasks with your colleagues, but we expect each of you to hand in your own, individual write-up. **You should not share your write-up or (even worse) your source code!**
- There will be three special assignments this semester. Each of them will be graded and the average grade will contribute 30% to your final grade.
- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer" or "justify intuitively", then a formal proof is **always** required.
- As with all exercises, the material covered in special assignments is relevant for the final exam.

Exercise 1 (Rigid k-coloring) *(50 points)*

A proper k-coloring of a graph is called *rigid*, if for every vertex of the graph all colors (except for its own) appear in its neighbourhood. In this exercise we look into algorithms that find a rigid k-coloring. Let n denote the number of vertices in the graph.

a) Provide a randomized algorithm that produces a rigid k-coloring for a graph in expected time $\mathcal{O}(k^{(1-1/k)n} \cdot \mathrm{poly}(n))$, provided it exists.

b) Provide a deterministic algorithm that produces a rigid k-coloring for a graph in time $\mathcal{O}((k-1)^n \cdot \mathrm{poly}(n))$, provided it exists.

c) For any $\epsilon \in (0, 1/k)$ provide a randomized algorithm that produces a rigid k-coloring for a graph in expected time $\mathcal{O}((k-1)^{(1-1/k+\epsilon)n} \cdot \mathrm{poly}(n))$, provided it exists and assuming the graph is connected.

   HINT: Construct a k-ary covering set of polynomial size. Prove that having the radius $(1-1/k+\epsilon)n$ works for this purpose and then use your algorithm from part b) appropriately. See lecture notes p. 155-156 for the definition and properties of k-ary covering codes. You can use the results from there. You might be able to relax the connectedness assumption somewhat, but having it may make things a little bit easier.

d) There is a way to decide whether a graph is k-colorable in time $\mathcal{O}(2^n \mathrm{poly}(n))$ as described by Björklund et al. [2009]. Briefly – and in your own words – describe the main components of their proof in the case of k-coloring. You don't need to read the whole paper to do this (but you can: it is really interesting and not too hard!) as the idea for the case of k-coloring is contained within a few pages. We could also try to use their ideas to decide the existence of a rigid k-coloring in comparable time, or at least in time better than just running the algorithms developed in parts a)-c). I don't know how to do it, but as a bonus question (for some bonus points) show how to decide rigid k-coloring relatively fast by using ideas from Björklund et al. [2009] or from somewhere else.

Exercise 2 (Properties of the cube) *(30 points)*

a) Show that the faces of the cube satisfy a strong Helly-type property: That is, let $\phi_1, \dots, \phi_m \in \{0, 1, *\}^n$ be faces of the n-dimensional Hamming cube. Show that if $\phi_i \cap \phi_j \neq \emptyset$ for all $1 \leq i, j \leq m$ then $\phi := \phi_1 \cap \dots \cap \phi_m \neq \emptyset$ and furthermore, ϕ itself is a face of the Hamming cube.

b) Let F be a CNF formula with $\Box \notin F$. Let the faces of the Hamming cube corresponding (in the sense of Chapter 5 of the lecture notes) to the clauses of F satisfy the properties from part a). Show that F is satisfiable.

c) Let $S \subseteq \{0, 1\}^n$ be nonempty. Show that there exists a tiling of the n-dimensional Hamming cube $\{0, 1\}^n$ into disjoint faces $\phi_1, \dots, \phi_{|S|}$ such that each ϕ_i contains exactly one element of S.

References

Andreas Björklund, Thore Husfeldt, and Mikko Koivisto. Set partitioning via inclusion-exclusion. *SIAM Journal on Computing*, 39(2):546–563, 2009.