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Introduction to Topological Data Analysis **Scribe Notes 2** **HS22**

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Topological Spaces, continued

Lemma 1. *Let* (X, T) *be some topological space, and* $Y \subseteq X$. *Then,* $U := \{A \cap Y \mid A \in T\}$ *is a topology on* Y. *We call this a* subspace topology.

Proof. We check the three conditions of a topology:

1. $\emptyset = \emptyset \cap Y$, therefore $\emptyset \in U$. Similarly, $Y = X \cap Y$, and thus $Y \in U$.
2. $\bigcup_{i \in I}(A_i \cap Y) = (\bigcup_{i \in I} A_i) \cap Y$, and thus $\bigcup_{i \in I}(A_i \cap Y) \in U$.
3. $\bigcap_{i=1}^{n}(A_i \cap Y) = (\bigcap_{i=1}^{n} A_i) \cap Y$, and thus $\bigcap_{i=1}^{n}(A_i \cap Y) \in U$.

□

Since we have seen that $\mathbb{R}^d$ is a topological space, this already tells us that all subsets of $\mathbb{R}^d$ are topological spaces.

Fact 2. *Let* X, Y *be two topological spaces. Then,* $X \times Y$ *is a topological space, with the so-called* product topology.

Definition 3. *A topological space* (X, T) *is* disconnected, *if there are two disjoint non-empy open sets* $U, V \in T$, *such that* $X = U \cup V$. *A topological space is* connected, *if it is not disconnected.*

Metric Spaces

Definition 4. *A* metric space (X, d) *is a set* X *of points and a distance function* $d : X \times X \to \mathbb{R}$ *satisfying*

1. $d(p, q) = 0$ *if and only if* $p = q$.
2. $d(p, q) = d(q, p)$, $\forall p, q \in X$. *(Symmetry)*
3. $d(p, q) \leq d(p, s) + d(s, q)$, $\forall p, q, s \in X$. *(Triangle inequality)*

Note that these three conditions imply that $d(p, q) \geq 0$ for all $p, q \in X$: If some distance $d(p, q)$ would be negative, we would have $0 = d(p, p) \leq d(p, q) + d(q, p) = 2 \cdot d(p, q) < 0$, a contradiction.

Fact 5. *Every metric space has a topology (the* metric space topology*) given by the open metric balls* $B(c, r) = \{p \in X \mid d(p, c) < r\}$ *and their unions.*

Maps between topological spaces

Definition 6. *A function* $f : X \to Y$ *is* continuous *if for every open set* $U \subseteq Y$*, its pre-image* $f^{-1}(U) \subseteq X$ *(the set of all elements* $x \in X$ *such that* $f(x) \in U$*) is open. Continuous functions are also called* maps*. If* f *is injective, it is called an* embedding*.*

Examples:

For $X \subseteq Y$, we write $X \hookrightarrow Y$ for the function $f(x) = x, \forall x \in X$. This function, which is also called the *inclusion map*, is continuous: $f^{-1}(U) = U \cap X$, which is open in the subspace topology on X.

For a function $f : \mathbb{R} \to \mathbb{R}$, continuity agrees with the "$\epsilon - \delta$" definition of continuity from calculus.

Definition 7. *A* homeomorphism *is a bijective map* $f : X \to Y$ *whose inverse is also continuous. Two topological spaces are* homeomorphic*, if there is a homeomorphism between them. We also write* $X \simeq Y$ *to say that* X, Y *are homeomorphic.*

Examples:

The boundary of a tetrahedron is homeomorphic to the sphere S^2. Idea: Take a point c within the tetrahedron, and send each point p to the point $f(p)$ on the ray from c through p such that $d(c, f(p)) = 1$.

$I := (-1, 1)$ is homeomorphic to $\mathbb{R}$. The following map f is a homeomorphism: $f : I \to \mathbb{R}$, $x \mapsto \frac{x}{1-|x|}$. Its inverse is $f^{-1} : \mathbb{R} \to I$, $y \mapsto \frac{y}{1+|y|}$.

All knots (embeddings of the circle into $\mathbb{R}^3$) are homeomorphic. Thus, we cannot distinguish between knots using only homeomorphism.

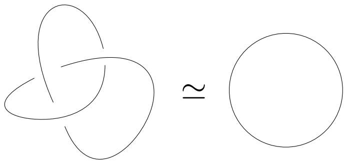

Figure 1: Two knots.

Definition 8. *An* isotopy *connecting* $X \subseteq \mathbb{R}^d$ *and* $Y \subseteq \mathbb{R}^d$ *is a continous map* $\phi : X \times [0,1] \to \mathbb{R}^d$, *such that* $\phi(X,0) = X$, $\phi(X,1) = Y$, *and* $\forall t \in [0,1]$, $\phi(\cdot,t)$ *is a homeomorphism between* X *and its image. Two spaces are called* isotopic, *if there is an isotopy connecting them.*

Examples:

Let $X \subset \mathbb{R}$ be the union of 0, and $[1,2]$, and let $Y \subset \mathbb{R}$ be the union of $[0,1]$ and 2. These spaces are homeomorphic ($X \simeq Y$), but not isotopic.

The two knots from Figure 1 above are also not isotopic.

We have also seen an isotopy between the two spaces in Figure 2, the isotopy is illustrated by the following video: https://www.youtube.com/watch?v=wDZx9B4TAXo

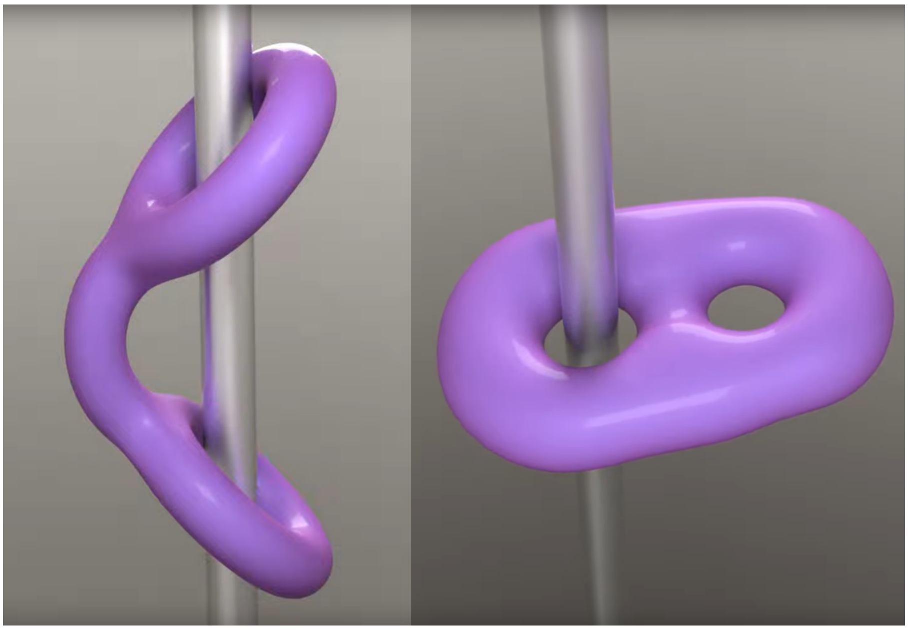

Figure 2: Left: Both handcuffs are connected to an infinite pole. Right: Only one loop of the handcuffs is connected to the infinite pole. These spaces are isotopic.

Definition 9. *Let* g, h *be maps* $X \to Y$*. A* homotopy *connecting* g *and* h *is a map* $H : X \times [0,1] \to Y$ *such that* $H(\cdot, 0) = g$ *and* $H(\cdot, 1) = h$*. In this case* g *and* h *are called* homotopic.

Examples:

The inclusion map $g : \mathbb{B}^3 \hookrightarrow \mathbb{R}^3$ (where $\mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$), and $h : \mathbb{B}^3 \to \mathbb{R}^3$ which sends every point to the origin, are homotopic, as shown by the homotopy

$$H(x, t) = (1 - t)g(x).$$

The identity function $g : S^1 \to S^1$ and $h : S^1 \to S^1$ which sends everything to a single point $p \in S^1$ are *not* homotopic.

Definition 10. *Two spaces* X, Y *are* homotopy equivalent *if there exist maps* $g : X \to Y$ *and* $h : Y \to X$ *such that:*

- $h \circ g$ *is homotopic to* id_X *(the identity map* $x \mapsto x$*), and*
- $g \circ h$ *is homotopic to* id_Y*.*

Example: The circle S^1 and $\mathbb{R}^2 \setminus \{0\}$ are homotopy equivalent. We pick g as the inlusion map $S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$, and $h(x) := \frac{x}{|x|}$. We see that $h \circ g(x) = x$, i.e., $h \circ g = \mathrm{id}_{S^1}$. Furthermore, $g \circ h(x) = h(x)$. $g \circ h$ and $\mathrm{id}_{\mathbb{R}^2 \setminus \{0\}}$ are homotopic as certified by the homotopy $H(x, t) := tx + (1 - t)h(x)$.

This example shows us that homotopy equivalence is a somewhat weaker property than homeomorphism, since these two spaces do not even have the same dimension (and thus cannot be homeomorphic), but they are homotopy equivalent.

Definition 11. *Let* $A \subseteq X$*. A* deformation retract *of* X *onto* A *is a map* $R : X \times [0,1] \to X$*, such that*

- $R(\cdot, 0) = \mathrm{id}_X$
- $R(x, 1) \in A, \forall x \in X$
- $R(a, t) = a, \forall a \in A, t \in [0,1]$

If such a deformation retract of X *onto* A *exists, we also say that* A is *a deformation retract of* X*.*

The intuition behind a deformation retract is to continuously shrink X to A, while leaving A fixed.

Fact 12. *If* A *is a deformation retract of* X *(there exists a deformation retract of* X *onto* A*), then* A *and* X *are homotopy equivalent.*

Examples:

The circle S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$: $R(x,t) = (1-t)x + t \cdot \frac{x}{|x|}$.

A punctured torus can be deformation retracted onto the symbol 8 where one of the two circles is rotated by 90, as seen by the following video:
https://www.youtube.com/watch?v=tz3QWrfPQj4

Lemma 13. *If* X *and* Y *are homeomorphic, they are also homotopy equivalent.*

Proof. Let $g : X \to Y$ be the homeomorphism, and $h := g^{-1}$ its inverse. Then $g \circ h = \mathrm{id}_Y$ and $h \circ g = \mathrm{id}_X$, and id is homotopic to itself. □

Fact 14. X, Y *are homotopy equivalent if and only if there exists a space* Z *such that* X *and* Y *are deformation retracts of* Z*.*

An example of this fact can be found in Figure 3 below:

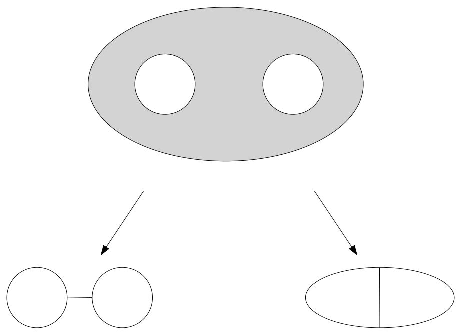

Figure 3: The top space deformation retracts to both spaces below, showing that they are homotopy equivalent.

We note that in general, showing existence of a map with certain properties (e.g., a homeomorphism, isotopy, homotopy) is easy, but showing that such a map cannot exist is hard. The idea of algebraic topology is to find invariant properties preserved by these maps: then, we know that no such map can exist between spaces that differ on these properties. An example of such an invariant property is the number of holes a space has, which we will formalize in the future.