ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Patrick Schnider

Introduction to Topological Data Analysis **Scribe Notes 4** **HS22**

Scribe notes by Simon Weber. Please contact me for corrections.
Lecture date: March 3, 2023
Last update: Friday 3rd March, 2023, 14:00

Simplicial Complexes

(Slides were used for this chapter, for figures refer to the slides)

Definition 1. *A* k*-simplex in* $\mathbb{R}^d$ *is the convex hull of* $k+1$ *affinely independent points in* $\mathbb{R}^d$*.*

A *face* of a simplex is the convex hull of a subset of its vertices. In particular, every face of a simplex is also a simplex. The empty set $\emptyset$ is also a face. The $(k-1)$-faces are called *facets*.

Definition 2. *A* geometric simplicial complex *is a finite family* K *of simplices such that*

- *if* $\tau \in K$ *and* σ *is a face of* τ*, then* $\sigma \in K$*, and*
- *for* $\sigma, \tau \in K$*, their intersection* $\sigma \cap \tau$ *is a face of both.*

We say the *dimension* of a simplicial complex is the maximum dimension of any simplex, and the dimension of a k-simplex is k.

Definition 3. *An* abstract simplicial complex K *is a family of subsets of a finite vertex set* $V(K)$ *such that if* $\tau \in K$ *and* $\sigma \subseteq \tau$*, then* $\sigma \in K$*.*

A k-simplex here is a subset of $k+1$ elements, and thus again called k-dimensional.

We can define from every geometric simplicial complex an abstract simplicial complex, by simply taking the set of points as the vertex set, and adding the correct subset for every simplex. In the inverse direction, we have to talk about geometric realizations:

Definition 4. *A geometric simplicial complex* K *is a geometric realization of some abstract simplicial complex* K'*, if there is an embedding* $e : V(K') \to \mathbb{R}^d$ *that takes every (abstract)* k*-simplex* $\{v_0, \ldots, v_k\}$ *in* K' *to the (geometric)* k*-simplex that is the convex hull of* $e(v_0), \ldots, e(v_k)$*.*

Does every abstract simplicial complex have a geometric realization? For 1-dimensional complexes (graphs), we know that dimension 3 suffices. This generalizes to the following theorem:

Theorem 5. *Every* k*-dimensional simplicial complex has a geometric realization in* $\mathbb{R}^{2k+1}$*.*

In fact, we can always find a geometric realization by placing the vertices as distinct points on the *moment curve* in $\mathbb{R}^{2k+1}$, which is the curve given by $f(t) = (t, t^2, \ldots, t^{2k+1})$.

Since we now know that abstract and geometric simplicial complexes can be translated into each other, we will never make the distinction between them again. As a subset of euclidean space, a simplicial complex thus also inherits the subspace topology from $\mathbb{R}^d$, which allows us to view simplicial complexes as topological spaces.

Definition 6. *A simplicial complex* K *is a* triangulation *of a topological space* X*, if* $|K|$ *is homeomorphic to* X*.*

Not all topological spaces are triangulable, but in this course we will not deal with these spaces. Also note that a triangulable space has infinitely many triangulations, for example by subdividing simplices.

Definition 7. *For a finite collection* U *of sets, its* nerve $N(U)$ *is a simplicial complex on the vertex set* U *that contains* $U_0, \ldots, U_k$ *as a* k*-simplex iff* $U_0 \cap \ldots \cap U_k \neq \emptyset$*.*

Definition 8. *Let* X *be a metric space, and* U *a finite family of closed subsets of* X*. We call* U *a* good cover*, if every non-empty intersection of sets in* U *is contractible (i.e., homotopy equivalent to a point).*

Theorem 9 (Nerve theorem). *If* U *is a good cover, then* $|N(U)|$ *is homotopy equivalent to* $\bigcup U$*.*

The nerve theorem fails if some sets in U are closed, and some open: We can have an open and a closed set which do not intersect, but whose union is connected.

Definition 10. *A map* $f : K_1 \to K_2$ *(which maps vertices of* K_1 *to vertices of* K_2*, also called a* vertex map*) is called* simplicial *if for every simplex* $\{v_0, \ldots, v_k\} \in K_1$*, we have that* $\{f(v_0), \ldots, f(v_k)\}$ *is a simplex in* K_2*.*

Fact 11. *Every continuous map* $f : |K_1| \to |K_2|$ *can be approximated arbitrarily closely by simplicial maps on appropriate subdivisions of* K_1 *and* K_2*.*

This shows that simplicial maps are the analogue of continuous maps in the world of simplicial complexes.

Definition 12. *Two simplicial maps* $f_1, f_2 : K_1 \to K_2$ *are* contiguous *if for every simplex* $\sigma \in K_1$ *we have that* $f_1(\sigma) \cup f_2(\sigma)$ *is a simplex in* K_2.

This is the analogue of two continuous maps being homotopic.

Definition 13. *A face of a simplicial complex is called* free*, if it is a non-maximal (not inclusion-maximal) and contained in a unique maximal face.*

Note that every face that is a superset of a free face is either a maximal face or also free.

Definition 14. *A* collapse *is the operation of removing all faces* γ *that contain some fixed free face* τ.

Definition 15. *A simplicial complex is* collapsible *if there is a sequence of collapses leading to a point.*

A collapse can be written as a deformation retract. Thus, a simplicial complex that is collapsible is contractible. We will see that the converse does not hold: A good counterexample for this is Bing's house with two rooms. In any triangulation of it, there are no free faces: As a 2-dimensional space, there are only vertices, edges and triangles. We only have to check edges, since triangles are maximal, and vertices are part of edges which are never maximal. Every edge is incident to at least two triangles (there are no edges on the "boundary"), and thus they are not free. Since we have no free faces, it is not collapsible.
But Bing's house is contractible. Why? It does not deformation retract to a point, but a 3-dimensional ball deformation retracts to both Bing's house and a point, see the slides for some figures.

The following table summarizes the equivalent words in "continuous topology" and in combinatorial topology on simplicial complexes:

"continuous" topology	combinatorial topology
topological spaces	simplicial complexes
continuous maps	simplicial maps
homotopic maps	contiguous maps
deformation retracts	collapses

For triangulable spaces, we can treat both sides as equivalent.

Homology

Chains

Let K be a simplicial complex with m_p p-simplices.

Definition 16. *A* p-chain c (in K) *is a formal sum*[1] *of* p*-simplices added with some coefficients from some ring* R*.*

$$c = \sum_{i=1}^{m_p} \alpha_i \sigma_i$$

where $\alpha_i \in R$ *and* $\sigma_i \in K$ *are* p*-simplices.*

Two p-chains $c = \sum \alpha_i \sigma_i$ and $c' = \sum \alpha_i' \sigma_i$ (both in K) can be added:

$$c + c' := \sum_{i=1}^{m_p} (\alpha_i + \alpha_i')\sigma_i$$

We write $C_p(K)$ for the set of all p-chains in K, called the p*-th chain group*. The following observation shows that this name makes sense:

Observation 17. $(C_p(K), +)$ *is an abelian group, it is free, and the* p*-simplices form a basis.*

Proof. To show that it is a group, we have

1. $\forall c_1, c_2 \in C_p(K)$, we have $c_1 + c_2 \in C_p(K)$
2. $\forall c_1, c_2, c_3 \in C_p(K)$,
   $(c_1 + c_2) + c_3 = \sum(\alpha_i^{(1)} + \alpha_i^{(2)})\sigma_i + \sum \alpha_i^{(3)}\sigma_i = \sum(\alpha_i^{(1)} + \alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i =$
   $\sum \alpha_i^{(1)}\sigma_i + \sum(\alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i = c_1 + (c_2 + c_3)$.
3. $0 = \sum 0\sigma_i \in C_p(K)$
4. $\forall c \in C_p(K)$ we have $-c = \sum(-\alpha_i\sigma_i) \in C_p(K)$ and $c + (-c) = \sum(\alpha_i - \alpha_i)\sigma_i = 0$

Commutativity follows from $+$ being commutative, thus the group is abelian. The p-simplices clearly form a basis, since the set of chains is defined as the set of formal sums of these p-simplices. □

[1] A formal sum just means that we formally write a sum, but that there is no meaning behind the operation of adding the simplices.