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Introduction to Topological Data Analysis **Scribe Notes 5** **FS23**

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Notes about the "dictionary" from scribe notes 4:
The connection between simplicial complexes and topological spaces is that every simplicial complex defines a topological space, since we can consider a geometric embedding, and the underlying space of the embedding inherits the subspace topology from $\mathbb{R}^d$. On the other hand, some topological spaces (the triangulable ones) can be expressed by simplicial complexes.
Between continuous maps and simplicial maps, a continuous map between the underlying spaces of simplicial complexes are simplicial maps. On the other hand, continuous maps between underlying spaces of simplicial complexes can be approximated by simplicial maps between subdivisions of the simplicial complexes.
A similar thing happens between homotopic maps and contiguous maps, as well as deformation retracts and collapses.
In general, we can say that the terms on the "combinatorial side" are special cases of the "continous" counterparts, and if we consider triangulable spaces, the continuous terms can be approximated in some way by their combinatorial counterparts, and can thus be considered to be the same.

Homology

Chains

Let K be a simplicial complex with m_p p-simplices.

Definition 1. *A* p-chain c (in K) *is a formal sum*[1] *of* p*-simplices added with some coefficients from some ring* R.

$$c = \sum_{i=1}^{m_p} \alpha_i \sigma_i$$

where $\alpha_i \in R$ *and* $\sigma_i \in K$ *are* p*-simplices.*

[1] A formal sum just means that we formally write a sum, but that there is no meaning behind the operation of adding the simplices.

Two p-chains $c = \sum \alpha_i \sigma_i$ and $c' = \sum \alpha'_i \sigma_i$ (both in K) can be added:

$$c + c' := \sum_{i=1}^{m_p} (\alpha_i + \alpha'_i)\sigma_i$$

We write $C_p(K)$ for the set of all p-chains in K, called the p-*th chain group*. The following observation shows that this name makes sense:

Observation 2. $(C_p(K), +)$ *is an abelian group, it is free, and the* p-*simplices form a basis.*

Proof. To show that it is a group, we have

1. $\forall c_1, c_2 \in C_p(K)$, we have $c_1 + c_2 \in C_p(K)$
2. $\forall c_1, c_2, c_3 \in C_p(K)$,
   $(c_1 + c_2) + c_3 = \sum(\alpha_i^{(1)} + \alpha_i^{(2)})\sigma_i + \sum \alpha_i^{(3)}\sigma_i = \sum(\alpha_i^{(1)} + \alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i =$
   $\sum \alpha_i^{(1)}\sigma_i + \sum(\alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i = c_1 + (c_2 + c_3)$.
3. $0 = \sum 0\sigma_i \in C_p(K)$
4. $\forall c \in C_p(K)$ we have $-c = \sum(-\alpha_i\sigma_i) \in C_p(K)$ and $c + (-c) = \sum(\alpha_i - \alpha_i)\sigma_i = 0$

Commutativity follows from + being commutative, thus the group is abelian. The p-simplices clearly form a basis, since the set of chains is defined as the set of formal sums of these p-simplices. □

Observation 3. *Equipped with the appropriate function* $\cdot : R \times C_p(K) \to C_p(K)$*,* $C_p(K)$ *is an* R*-module.*

Proof. The proof is similar and left as an exercise, but the statement should feel natural since every chain is simply described by a vector of m_p elements of R, with addition being element-wise addition in R. □

From now on we will always work with the ring $R = \mathbb{Z}_2$, so in particular we have that $c + c = 0$.

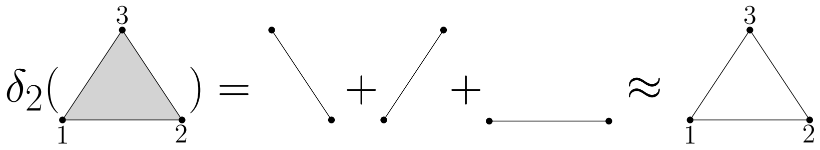
Boundaries

Let $\sigma = \{v_0, \dots, v_p\}$ be a p-simplex. Then, $\delta_p(\sigma)$ is defined by

$$\{v_1, \dots, v_p\} + \{v_0, v_2, \dots, v_p\} + \dots + \{v_0, \dots, v_{p-1}\} = \sum_{i=0}^{p} \{v_0, \dots, \hat{v_i}, \dots, v_p\}$$

In the above notation, $\hat{v_i}$ denotes that the element v_i is omitted from the set. Note that $\delta_p(\sigma)$ is a $(p-1)$-chain.

Examples:

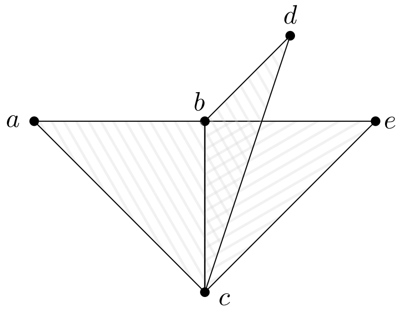


$$\delta_0(\,\cdot\,) = 0$$

We have seen that δ_p is a map that sends a p-simplex to a $(p-1)$-chain. Extending this to work on chains, δ_p defines a *boundary operator* homomorphism:

$$\delta_p : C_p(K) \to C_{p-1}(K)$$
$$c = \sum \alpha_i \sigma_i \mapsto \delta_p(c) = \sum \alpha_i(\delta_p(\sigma_i))$$

Let us apply this definition to the following example:



$$\begin{aligned}\delta_2(abc + bcd) &= \delta_2(abc) + \delta_2(bcd) \\ &= (ab + bc + ac) + (bc + cd + bd) \\ &= ab + ac + cd + bd\end{aligned}$$

$$\begin{aligned}\delta_2(abc + bcd + bce) &= (ab + bc + ac) + (bc + cd + bd) + (bc + ce + be) \\ &= ab + bc + ac + cd + bd + ce + be\end{aligned}$$

Lemma 4. *For* $p > 0$, $\delta_{p-1} \circ \delta_p(c) = 0$, *for any* p*-chain* c.

In other words, the boundary of a boundary is empty.

In the example above, $\delta_1(ab+ac+cd+bd) = (a+b)+(a+c)+(c+d)+(b+d) = 0$.

Proof. It is enough to show this for simplices, as $\delta_{p-1} \circ \delta_p(c) = \delta_{p-1}(\sum \alpha_i(\delta_p(\sigma_i))) = \sum \alpha_i(\delta_{p-1} \circ \delta_p(\sigma_i))$.
For a p-simplex σ, every $(p-2)$-face is contained in exactly 2 $(p-1)$-faces, and does thus not appear in $\delta_{p-1} \circ \delta_p(\sigma)$. □