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Introduction to Topological Data Analysis — Scribe Notes 6 — FS23

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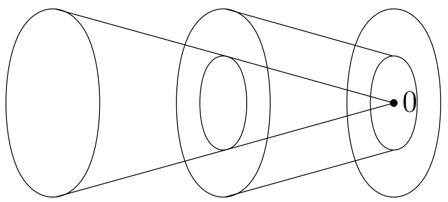
For a k-dimensional simplicial complex K, we get a sequence of homomorphisms, called the *chain complex*:

$$0 = C_{k+1}(K) \xrightarrow{\delta_{k+1}} C_k(K) \xrightarrow{\delta_k} C_{k-1}(K) \xrightarrow{\delta_{k-1}} C_{k-2}(K) \cdots C_2(K) \xrightarrow{\delta_2} C_1(K) \xrightarrow{\delta_1} C_0(K) \xrightarrow{\delta_0} C_{-1} = 0$$



Cycle and boundary groups

Definition 1. *A* p*-chain* c *is a* p-cycle *if* $\delta(c) = 0$.
Z_p *is the* p-th cycle group, *consisting of all* p*-cycles.*

Lemma 2. Z_p *is a group.*

Proof. $Z_p = \ker \delta_p$. (Recall that the kernel of a homomorphism is a subgroup of its domain.) □

Definition 3. *A* p*-chain* c *is a* p-boundary *if* $\exists c' \in C_{p+1}$ *such that* $\delta(c') = c$.
B_p *is the* p-th boundary group, *consisting of all* p*-boundaries.*

Lemma 4. B_p *is a group.*

Proof. $B_p = \operatorname{im} \delta_{p+1}$. □

Fact 5. $B_p \subseteq Z_p \subseteq C_p$, *and all of them are abelian and free.*

We will not prove this statement here, but to see that $B_p \subseteq Z_p$, consider that the boundary of a boundary is empty (Lemma 4 from the scribe notes 5).

Homology

Definition 6. *The* p-th homology group $H_p(K; \mathbb{Z}_2)$ *is the quotient group* $Z_p(K)/B_p(K)$.

In essence, in this group cycles that differ only by boundaries are equivalent. More formally, the coset $[c] = c + B_p$ is the *homology class* of c. We say that c and c' are *homologous*, if $[c] = [c']$, or equivalently $c \in c' + B_p$ or equivalently $c + c' \in B_p$.

Example:

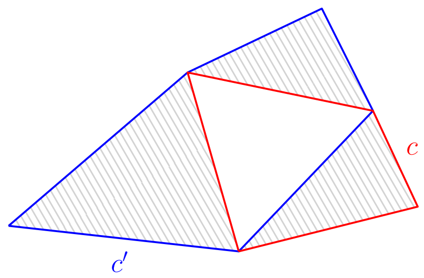


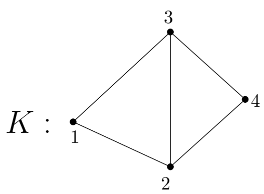
Figure 1: c' and c are homologous cycles.

Fact 7.

- H_p *is abelian and free.*
- H_p *is a* $\mathbb{Z}_2$*-vector space.*

Remark: Homology can be defined over any ring, e.g. over $\mathbb{Z}$ instead of $\mathbb{Z}_2$, but in this case it might not be free.

Example of the Homology group:



$$H_1(K) : \{0, 123, 234, 1234\} \cong \mathbb{Z}_2^2$$

Definition 8. $\beta_p := \dim H_p = \dim Z_p - \dim B_p$ *is the* p-th Betti number.

In the defintion above, dim denotes the dimension of a vector space as you know it from Linear Algebra, i.e., $\dim H_p$ is the number of elements in a basis of H_p.

We can now also talk about homology of triangulable topological spaces; we can simply triangulate them and talk about the homology of the triangulation. But, so far, it seems like the structure of the homology group might differ depending on the choice of triangulation of some topological space. The aim of the following chapter is to remove this dependency.

Singular Homology

The idea of singular homology is to remove the need for a fixed triangulation by looking at all possible simplices at once.

Let X be a topological space, and let Δ^p be the standard p-simplex in $\mathbb{R}^{p+1}$.

Definition 9. *A* singular p-simplex *is a map* $\sigma : \Delta^p \to X$.

Note that in this definition we do not require σ to be injective, thus it would even be possible to map the simplex to a single point.

We now define C_p the same way as before, but now on the family of all singular p-simplices, which makes the group uncountably infinite. We also define δ_p as before, leading to Z_p and B_p now also being uncountably infinite. Similarly, $H_p(X) = Z_p(X)/B_p(X)$.

Theorem 10. *Let X be a topological space, K a triangulation of X. Then we have $H_p(X) \cong H_p(K)$ for all $p \geq 0$.*

Corollary 11. *Let K_1, K_2 be two distinct triangulations of X. Then, $H_p(K_1) \cong H_p(K_2)$ for all $p \geq 0$, that is, homology is independent of the chosen triangulation.*

The 0-th homology group

Recall that the 0-simplices of a simplicial complex K are simply its vertices. Since the vertices do not have any boundaries, every vertex is a 0-cycle. The boundary of a 1-simplex simply consists of the 2 vertices which are connected by the edge.

We can thus see that two vertices v_1 and v_2 are homologous if there is a path from v_1 to v_2, and the homology class $[v_1]$ is simply the connected component containing v_1.

Observation 12. *$\beta_0(K)$ is the number of connected components of K.*

We can also conclude that the 0-homology classes are formal sums of connected components.

Homology of Spheres

We will now investigate the homology of the spheres S^d. We will prove the following theorem, as our intuition would tell us:

Theorem 13. *For any* $d > 0$*, we have*

$$H_p(S^d) = \begin{cases} \mathbb{Z}_2 & p \in \{0, d\} \\ 0 & \textit{else.} \end{cases}$$

$$\beta_p(S^d) = \begin{cases} 1 & p \in \{0, d\} \\ 0 & \textit{else.} \end{cases}$$

Since we have shown in the chapter "Singular Homology" that homology is independent from the chosen triangulation, let us fix some triangulation of the sphere S^d. A good candidate (for its simplicity) is $S^d \simeq \delta(\Delta^{d+1})$, on the vertex set $V = \{v_0, \ldots, v_{d+1}\}$.

$H_0(S^d)$: Let us first investigate $H_0(S^d)$. Since all vertices are connected, all vertices are homologous, and $H_0(S^d) = \langle [v] \rangle \cong \mathbb{Z}_2$.

$H_d(S^d)$: Finally, let us check $H_d(S^d)$. Since $\delta(\Delta^{d+1})$ is a d-dimensional simplicial complex, and thus does not contain any $(d+1)$-simplices, $\delta_{d+1} : C_{d+1} \to C_d$ maps everything (the only element is 0, anyways) to 0. Thus, $H_d(S^d) \cong Z_d(S^d)$.

There is exactly one non-trivial cycle in $\delta(\Delta^{d+1})$, the one containing all d-simplices. Thus, $H_d(S^d) \cong Z_d(S^d) \cong \mathbb{Z}_2$.