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Introduction to Topological Data Analysis **Scribe Notes 7** **FS23**

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Homology of Spheres

We will now investigate the homology of the spheres S^d. Since we have shown in the chapter "Singular Homology" that homology is independent from the chosen triangulation, let us fix some triangulation of the sphere S^d. A good candidate (for its simplicity) is $S^d \simeq \delta(\Delta^{d+1})$, on the vertex set $V = \{v_0, \ldots, v_{d+1}\}$.

$H_0(S^d)$: Let us first investigate $H_0(S^d)$. Since all vertices are connected, all vertices are homologous, and $H_0(S^d) = \langle [v] \rangle \cong \mathbb{Z}_2$.

$H_d(S^d)$: Now, let us check $H_d(S^d)$. We first compute Z_d: Obviously, the zero element is part of Z_d. Furthermore, the d-simplices are exactly the sets $\sigma_i = \{v_0, \ldots, \hat{v_i}, \ldots, v_{d+1}$. The sum c of all these d-simplices must be a cycle, since every $d-1$-simplex occurs in exactly two d-simplices, thus the boundary of c must be empty. Thus, $c \in Z_d$. We cannot have any other cycle, since for any other chain there must be some d-simplex for which we include one neighbor but not the other, thus this d-simplex would be part of the boundary. We conclude that $Z_d(S^d) = \langle [c] \rangle$.

Since $\delta(\Delta^{d+1})$ is a d-dimensional simplicial complex, and thus does not contain any $(d+1)$-simplices, c cannot be a boundary. Since B_d is a subgroup of Z_d, we thus get that $B_d(S^d)$ is the group containing only 0. Alternatively, we can also get this by noticing that $C_{d+1} = 0$, and $B_d = \operatorname{im} \delta_{d+1} = 0$.

We finally get $H_d(S^d) = Z_d/B_d = Z_d \cong \mathbb{Z}_2$.

$H_p(S^d)$: Finally, let us go to $H_p(S^d)$, for $0 < p < d$: Let $c = \sum \alpha_i \sigma_i$ be a p-cycle. We aim to show that c is homologous to the 0-chain, i.e., that $[c] = 0$. Equivalently, we show that c must be a boundary.

Let $\sigma = (v_{m_0}, \ldots, v_{m_p})$ be any p-simplex in c which does not include v_0. We will keep replacing such simplices by simplices which do contain v_0, until we have no more simplices not containing v_0.

Let b be the $(p+1)$-simplex $(v_0, v_{m_0}, \dots, v_{m_p})$. Note that $b \in \delta(\Delta^{d+1})$ and thus $\delta(b)$ is a p-boundary. Also note that σ is in $\delta(b)$. Furthermore, σ is the only p-simplex in $\delta(b)$ which does not contain v_0. We now add $\delta(b)$ to c, to get $c' := c + \delta(b)$. Since we added a boundary, $[c] = [c']$ (i.e., c and c' are homologous). Furthermore, c' contains one fewer p-simplex not containing v_0, when compared to c.

We repeat this process until we reach a cycle c^* in which every p-simplex contains v_0. We now claim that c^* must be the trivial cycle (0): Assume c^* contains some p-simplex $a = (v_0, v_{a_1}, \dots, v_{a_p})$. Then, the $(p-1)$-simplex $a' = (v_{a_1}, \dots, v_{a_p})$ is part of $\delta(a)$. But, a' cannot be part of the boundary of any other p-simplex in c^*, since the only p-simplex containing a' as a face while also containing v_0 is a. Thus, to have an empty boundary, c^* must be 0. We thus have $[c^*] = 0$, and by construction, $[c] = [c^*]$, therefore $[c] = 0$ as we aimed to prove.

We have proven that every cycle is homologous to 0, and we can conclude that for all $0 < p < d$, $H_p(S^d) = 0$.

By these arguments we conclude the following theorem:

Theorem 1. *For any $d > 0$, we have*

$$H_p(S^d) = \begin{cases} \mathbb{Z}_2 & p \in \{0, d\} \\ 0 & \textit{else.} \end{cases}$$

$$\beta_p(S^d) = \begin{cases} 1 & p \in \{0, d\} \\ 0 & \textit{else.} \end{cases}$$