ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Patrick Schnider

Introduction to Topological Data Analysis — Scribe Notes 8 — FS23

Scribe notes by Simon Weber. Please contact me for corrections.
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Induced Homology

Let $f : K_1 \to K_2$ be a simplicial map. This induces a *chain map*

$$f_\# : C_p(K_1) \to C_p(K_2)$$
$$c = \sum \alpha_i \sigma_i \mapsto f_\#(c) := \sum \alpha_i \tau_i, \text{ where } \tau_i = \begin{cases} f(\sigma_i) & \text{if } f(\sigma_i) \text{ is p-simplex in } K_2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(\sigma_i)$ is always a simplex in K_2 since f is a simplicial map, but it could be a smaller simplex. This is why we have the condition in the above definition of τ_i.

We have:

- $f_\# \circ \delta = \delta \circ f_\#$
- $f_\#(B_p(K_1)) \subseteq f_\#(Z_p(K_1))$
- $f_\#(Z_p(K_1)) \subseteq Z_p(K_2)$, $f_\#(B_p(K_1)) \subseteq B_p(K_2)$

From this chain map $f_\#$, we further get a well-defined *induced homomorphism* between the homology groups of K_1 and K_2:

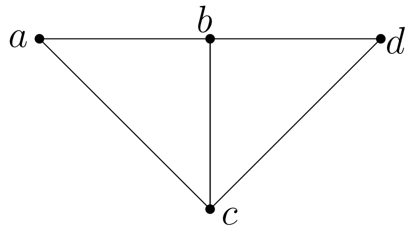
$$f_* : H_p(K_1) \to H_p(K_2)$$
$$[c] = c + B_p \mapsto f_\#(c) + B_p(K_2) = [f_\#(c)]$$

Fact 1. *If $H_p(K_1)$ and $H_p(K_2)$ are vector spaces (as they are in e.g. $\mathbb{Z}_2$-homology, which is what we are using), then f_* is a linear map.*

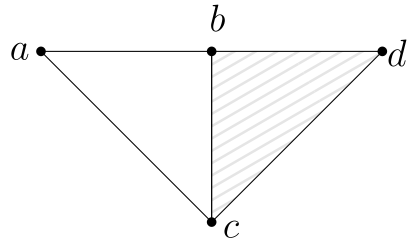
We also get that if we have $f : X \to Y$, $g : Y \to Z$, $g \circ f : X \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Example:

K_1



K_2



b

We consider $f : K_1 \hookrightarrow K_2$ the inclusion map.

$$H_1(K_1) = \{0, [abc], [bcd], [abdc]\} \cong \mathbb{Z}_2^2$$

$$f_*(0) = 0, \; f_*([abc]) = [abc]$$
$$f_*([bcd]) = 0, \; f_*([abdc]) = [abc]$$

Fact 2. *If* $f, g : K_1 \to K_2$ *are contiguous,* $f_* = g_*$.

Note that the definition of induced homology extends from simplicial maps to maps between any topological spaces. Since a map f must take cycles to cycles and boundaries to boundaries, it also defines a map f_* between the homology groups of its domain and codomain. We will not state the exact definitions, but the following fact is the continuous analogue (remember that two simplicial maps being contiguous is analogous to two maps being homotopic) of the previous fact.

Fact 3. *If* $f, g : X \to Y$ *are homotopic,* $f_* = g_*$.

Corollary 4. *If* $f : X \to Y$ *is a homotopy equivalence (i.e., there exists* $g : Y \to X$ *such that* f, g *prove homotopy equivalence of* X *and* Y*), then* f_* *is an isomorphism.*

Corollary 5. *If* X *is contractible,* $H_p(X) = \begin{cases} \mathbb{Z}_2 & p = 0, \\ 0 & \textit{otherwise.} \end{cases}$

Application: Brouwer fixed point theorem

Theorem 6 (Brouwer fixed point theorem). *Let* $f : \mathbb{B}^d \to \mathbb{B}^d$ *be continuous. Then,* f *has a fixed point, that is,* $\exists x \in \mathbb{B}^d$ *such that* $f(x) = x$*.*

This theorem has many fascinating implications:

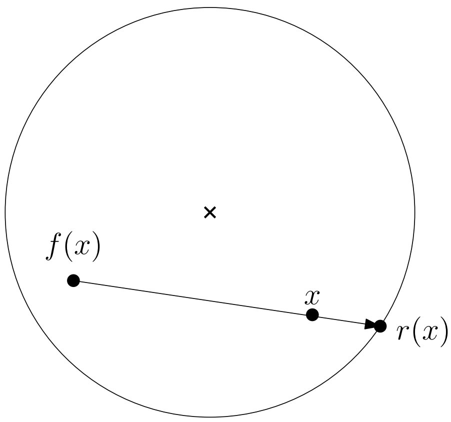
- Take two sheets of paper lying on top of each other. Crumple the top paper and set it back onto the other paper. No matter how you crumpled the paper, at least one point of the crumpled paper lies exactly above its corresponding point in the bottom paper.
- If you open a map of Switzerland in Switzerland, there is at least one point on the map which is at its exact position.
- If you take a cup of liquid and stir or slosh it, at least one atom ends up at its original position (but if you shake you might break continuity).
- The theorem also has many applications in mathematics and computer science, such as in fair divisions or for proving existence of Nash equilibria.

To prove Theorem 5, we first introduce the following definition and a helper lemma, which we only prove after proving Theorem 5 itself.

Definition 7. *A map* $r : X \to A \subseteq X$ *is a* retraction *if* $r(a) = a$*,* $\forall a \in A$*.*

Lemma 8. *There is no retraction* $r : \mathbb{B}^d \to S^{d-1}$*.*

Proof of Theorem 5. Proof by contradiction: Assume $f : \mathbb{B}^d \to \mathbb{B}^d$ has no fixed point. For each x, consider the ray $\overrightarrow{f(x)x}$ and let $r(x)$ be the intersection of this ray with S^{d-1}. Then, $r : \mathbb{B}^d \to S^{d-1}$ is continuous (which we do not prove here) and $r(s) = s \ \forall s \in S^{d-1}$, since no matter where $f(s)$ lies, $\overrightarrow{f(s)s}$ first intersects S^{d-1} in s. Thus, r is a retraction, which does not exist by Lemma 7. □



Proof of Lemma 7. Consider i, the inclusion map $S^{d-1} \hookrightarrow \mathbb{B}^d$, and a retraction $r : \mathbb{B}^d \to S^{d-1}$.

By definition, we have $r \circ i = \mathrm{id}$. Let us look at the induced maps of r and i in the $(d-1)$-th homologies of S^{d-1} and $\mathbb{B}^d$. Recall that $H_{d-1}(S^{d-1}) \cong \mathbb{Z}_2$ and $H_{d-1}(\mathbb{B}^d) \cong 0$. We thus view i_* as a homomorphism from $\mathbb{Z}_2$ to 0, and r_* as a homomorphism from 0 to $\mathbb{Z}_2$. But since $r \circ i = \mathrm{id}$, we also have $r_* i_* = \mathrm{id}$. We can combine this to reach a contradiction:

$$1 = \mathrm{id}(1) = r_* \circ i_*(1) = r_*(0) = 0$$

Thus, either i or r cannot exist, but since i exists, r cannot. □

$$\begin{array}{ccccc}
 & & \overset{id}{\curvearrowright} & & \\
H_{d-1}(S^{d+1}) & \overset{i_*}{\to} & H_{d-1}(\mathbb{B}^d) & \overset{r_*}{\to} & H_{d-1}(S^{d-1}) \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{Z}_2 & \overset{i_*}{\to} & 0 & \overset{r_*}{\to} & \mathbb{Z}_2 \\
 & & \underset{id}{\smile} & &
\end{array}$$