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Introduction to Topological Data Analysis **Scribe Notes 9** **FS23**

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A filtration is a nested sequence of subspaces:

$$\mathcal{F} : X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n = X$$

For each $i \leq j$, we have the inclusion map $\iota_{i,j} : X_i \hookrightarrow X_j$.

Given these functions ι, we get induced maps in homology: $h_p^{i,j} = \iota_* : H_p(X_i) \to H_p(X_j)$.

Given a function $f : X \to \mathbb{R}$, we can define the (uncountably infinite) *sublevel set filtration* $X_a = f^{-1}(-\infty, a]$.

A simplicial filtration is a nested sequence of subcomplexes:

$$\mathcal{F} : K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K$$

We call a simplicial filtration *simplex-wise*, if $K_i \setminus K_{i-1}$ is a single simplex (or empty).

We call a function $f : K \to \mathbb{R}$ *simplex-wise monotone* if for every $\sigma \subseteq \tau$ we have $f(\sigma) \leq f(\tau)$. A simplex-wise monotone function guarantees us that the sublevel set filtration by f gives a proper simplicial filtration. Note that it does not necessarily guarantee us that the sublevel set filtration is simplex-wise (e.g., consider a function f that is not injective).

We can also define a simplicial filtration by ordering our vertices $v_0, v_1, \ldots, v_n$. Then, let K_i be the simplicial complex induced by the vertices $v_0, \ldots, v_i$. Then, we call the simplices $K_i \setminus K_{i-1}$ added when adding v_i the *lower star* of v_i. Thus, this type of filtration is also called the *lower star filtration.*

Definition 1. *Let* (M, d) *be a metric space. Let* P *be a finite subset of* M*, and* $r > 0$ *a real number. The* Čech complex $\mathbb{C}^r(P)$ *is the nerve of the family of balls* $B(p, r) = \{x \in M | d(p, x) \leq r\}$ *for all* $p \in P$.

Since the balls $B(p, r)$ form a good cover, the nerve theorem tells us that the Čech complex is homotopy equivalent to the union of the balls.

By looking at the sequence of Čech complexes for increasing r, we get a simplicial filtration.

Definition 2. *The* p-th persistent homology group $H_p^{i,j}$ *is defined by*

$$H_p^{i,j} := \operatorname{im} h_p^{i,j} = Z_p(K_i)/(B_p(K_j) \cap Z_p(K_i)).$$

This definition characterizes the cycles that that are present already in K_i and that are not boundaries even in K_j.

Definition 3. *The* p-th persistent Betti numbers $\beta_p^{i,j}$ *are the dimensions of the* p*-th persistent homology groups:* $\beta_p^{i,j} = \dim H_p^{i,j}$.