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Introduction to Topological Data Analysis **Scribe Notes 11&14** **FS23**

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Simplicial Complexes on Point Sets

Definition 1. *Given a metric space* (M, d)*, a finite point set* $P \subseteq M$*, and a real number radius* $r > 0$*, the* Čech complex $\mathbb{C}^r(P)$ *is defined as the nerve of the set of balls*

$$B(p, r) = \{x \in M \mid d(p, x) \leq r\}$$

for all $p \in P$.

The Čech complex has the nice property that by the nerve theorem, it is homotopy equivalent to the union of the balls $B(p, r)$. Sadly, checking whether a large number of balls have a common intersection can be computationally expensive.

Definition 2. *Given a metric space* (M, d)*, a finite point set* $P \subseteq M$*, and a real number radius* $r > 0$*, the* Vietoris-Rips complex $\mathbb{VR}^r(P)$ *is defined as the simplicial complex containing a simplex* σ *if and only if* $d(p, q) \leq 2r$ *for every pair* $p, q \in \sigma$.

Clearly, by definition, the Čech complex and the Vietoris-Rips complex for the same radius and the same point set have the same set of 1-simplices (the same 1-skeleton). While the Čech complex then contains additional information about the common intersections of balls, the Vietoris-Rips complex is simply the clique complex of this 1-skeleton. This makes the Vietoris-Rips complex easier to compute. Furthermore, we make the following simple observation:

Observation 3. $\mathbb{C}^r(P) \subseteq \mathbb{VR}^r(P) \subseteq \mathbb{C}^{2r}(P)$.

For large enough radii, both the Čech and the Vietoris-Rips complex become complete, and thus contain 2^n simplices. We thus next look at the so-called Delaunay triangulation, which only has complexity $O(n^{\lceil d/2 \rceil})$.

Definition 4. *Given a finite point set* $P \subset \mathbb{R}^d$*, a* Delaunay simplex *is a geometric simplex whose vertices are in* P *and lie on the boundary of a ball whose interior contains no points of* P*.*

A Delaunay triangulation $\mathrm{Del}(P)$ *of* P *is a geometric simplicial complex where every simplex is a Delaunay simplex and whose underlying space covers the convex hull of* P*.*

Definition 5. *Given a finite point set* $P \subset \mathbb{R}^d$*, the* extended Delaunay complex *is the simplicial complex where for every face* σ*, for* $d' \leq d$*, every* d'*-face of* σ *is a Delaunay simplex.*

It is a well-known fact that for a point set in general position (no $d + 2$ points lie on a common sphere), there is a unique Delaunay triangulation. Furthermore, in this case the extended Delaunay complex and the unique Delaunay triangulation coincide.

Definition 6. *Given a finite point set* $P \subset \mathbb{R}^d$*, the* Voronoi diagram *is the tessellation of* $\mathbb{R}^d$ *into the* Voronoi cells

$$V_p = \{x \in \mathbb{R}^d \mid d(x,p) \leq d(x,q) \forall q \in P\}$$

for all $p \in P$*.*

Fact 7. *The nerve of the Voronoi cells of* P *is the extended Delaunay complex of* P*.*

Based on the Delaunay triangulation, we define the *Alpha complex* by parameterizing using a radius as follows:

Definition 8. *Given a finite point set* $P \subset \mathbb{R}^d$ *in general position as well as a real number radius* $r > 0$*, the* Alpha complex $\mathrm{Del}^r(P)$ *consists of all simplices* $\sigma \in \mathrm{Del}(P)$ *for which the circumscribing ball of* σ *has radius at most* r*.*

The following fact provides us with an alternative definition of the Alpha complex:

Fact 9. *The Alpha complex* $\mathrm{Del}^r(P)$ *is the nerve of the sets* $B(p,r) \cap V_p$ *for all* $p \in P$*.*

Since the Alpha complex is a subset of the Delaunay triangulation (and for large enough radius is equal to the Delaunay triangulation), it also has complexity $O(n^{\lceil d/2 \rceil})$.

Subsample Complexes

Definition 10. *Given a finite point set* Q *and a point set* $P \supset Q$ *in some metric space, we say that a simplex* $\sigma \subseteq Q$ *is* weakly witnessed *by* $x \in P \setminus Q$*, if* $d(q, x) \leq d(p, x)$ *for every* $q \in \sigma$ *and* $p \in Q \setminus \sigma$.

Note that the set of weakly witnessed simplices is not downwards closed. We thus define a simplicial complex by requiring that all faces are weakly witnessed:

Definition 11. *The* Witness complex $\mathbb{W}(Q, P)$ *is the collection of simplices on* Q *for which all faces are weakly witnessed by some point* $p \in P \setminus Q$.

Note that if we take the metric space $\mathbb{R}^d$ and we let P be the whole $\mathbb{R}^d$, then $\mathbb{W}(Q, P) = Del(Q)$, and by definition we thus get in general that $\mathbb{W}(Q, P) \subseteq Del(Q)$.

To arrive at a filtration, we again have to introduce a parameter $r > 0$:

Definition 12. *Given a finite point set* Q *and a point set* $P \supset Q$ *in some metric space as well as a real number radius* $r > 0$*, the* parameterized *Witness complex* $\mathbb{W}^r(Q, P)$ *is defined as follows:*
An edge pq *is in* $\mathbb{W}^r(Q, P)$ *if it is weakly witnessed by* $x \in P \setminus Q$ *and* $d(p, x) \leq r$ *and* $d(q, x) \leq r$. *A simplex* σ *is in* $\mathbb{W}^r(Q, P)$ *if all its edges are.*

The idea of this complex is that it should approximate the Vietoris-Rips complex on P. There are theoretical guarantees about this approximation for manifolds of dimension at most 2, but the parameterized witness complex may fail to capture the topology of manifolds in dimension 3 and above.

Note that from the definition it is not guaranteed that the parameterized Witness complex is a subcomplex of the Witness complex.

Definition 13. *Given two finite point sets* Q, P *in* $\mathbb{R}^d$*, as well as a graph* $G(P)$ *with vertices in* P*, we define* $\nu : P \to Q$ *by sending each point in* P *to its closest point in* Q. *The* graph induced complex $\mathbb{G}(Q, G(P))$ *contains a simplex* $\sigma = \{q_0, \dots, q_k\} \subset Q$ *if and only if there is a clique* $\{p_0, \dots, p_k\}$ *in* $G(P)$ *for which* $\nu(p_i) = q_i$.

We again parameterize this:

Definition 14. *Let* $G^r(P)$ *be the graph on* P *where* pq *is an edge if and only if* $d(p, q) \leq 2r$. *The* parameterized graph induced complex $\mathbb{G}^r(Q, P)$ *is defined as* $\mathbb{G}(Q, G^r(P))$.

This complex again has theoretical guarantees of approximating the Vietoris-Rips complex on $P \cup Q$.