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Introduction to Topological Data Analysis **Scribe Notes 20** **FS23**

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Distances for Reeb Graphs

Interleaving Distance

When do we want two Reeb graphs to be considered the same, and thus have distance 0? We definitely need that the graphs are isomorphic in the sense of graph isomorphism. But further than that, we also want that this graph isomorphism is also "function preserving". In other words, the critical points should lie on the same function levels. The idea of the interleaving distance is to measure how far away from this we are. Thus, given two Reeb graphs R_f, R_g, "how much" is missing towards a "function preserving isomorphism"? Towards formalizing this idea, we need a few definitions.

Note that when we compare two Reeb graphs R_f, R_g, those can be Reeb graphs of *different spaces* with regards to *different functions*.

Definition 1. *An* ϵ-thickening X_ϵ *of some topological space* X *is given by* $X_\epsilon := X \times [-\epsilon, +\epsilon]$.

Definition 2. *For a Reeb graph* R_f *consider a function* $f_\epsilon : (R_f)_\epsilon \to \mathbb{R}$ *such that*

$$(x, t) \mapsto f(x) + t.$$

The ϵ-smoothing *of* R_f*, denoted by* $S_\epsilon(R_f)$ *is the Reeb graph of* $(R_f)_\epsilon$ *with regards to* f_ϵ.

Note that when we say $(R_f)_\epsilon$, we mean an ϵ-thickening of R_f, *not* a Reeb graph with regards to some function f_ϵ. The ϵ-smoothing $S_\epsilon(R_f)$ is then a Reeb graph with regards to the function f_ϵ, but *of* $(R_f)_\epsilon$, and not of the original space R_f is the Reeb graph of.

Furthermore, when we write $f(x)$ for some $x \in R_f$, we mean that we extend f to some function $f^* : R_f \to \mathbb{R}$ by defining $f^*(x) = f(f^-1(x))$. We will just call this function f as well for simplicity.

An example of these definitions can be seen in Figure 1.

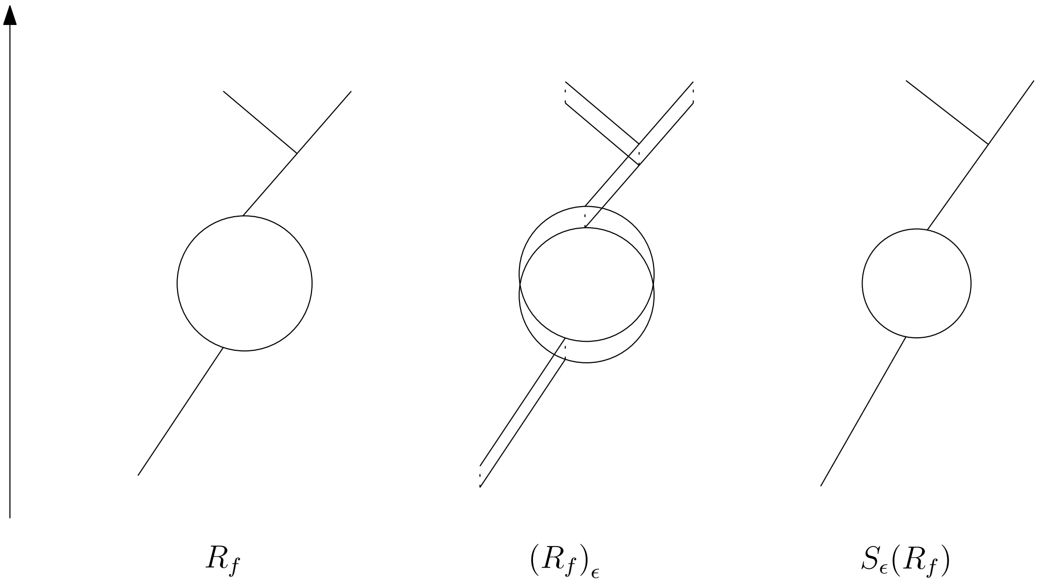

Figure 1: A Reeb graph, its ϵ-thickening, and its ϵ-smoothing.

Definition 3. *The function* $\iota : R_f \to S_\epsilon(R_f)$ *with* $x \mapsto [(x, 0)]$ *is the* quotiented inclusion map. *Here,* $[(x, 0)]$ *denotes the equivalence class, or the connected component that contains* $(x, 0)$ *in* $f_\epsilon^{-1}(f_\epsilon(x, 0))$.

Consider some function $\mu : R_f \to R_g$ which is function preserving, i.e., $f(x) = g(\mu(x))$ for all $x \in R_f$. A function-preserving map $\mu : R_f \to S_\epsilon(R_g)$ induces a function preserving map $\mu_\epsilon : S_\epsilon(R_f) \to S_{2\epsilon}(R_g)$ with $[x, t] \mapsto [\mu(x), t]$.

Definition 4 (Reeb graph interleaving). *A Reeb graph interleaving is a pair of function preserving maps* $\varphi : R_f \to S_\epsilon(R_g)$, $\psi : R_g \to S_\epsilon(R_f)$ *are* ϵ*-interleaved, if the following diagram commutes:*

$$\begin{array}{ccccc} R_f & \xrightarrow{\iota} & S_\epsilon(R_f) & \xrightarrow{\iota_\epsilon} & S_{2\epsilon}(R_f) \\ & \overset{\psi}{\nwarrow} \; \overset{\varphi}{\searrow} & & \overset{\psi_\epsilon}{\nwarrow} \; \overset{\varphi_\epsilon}{\searrow} & \\ R_g & \xrightarrow{\iota} & S_\epsilon(R_g) & \xrightarrow{\iota_\epsilon} & S_{2\epsilon}(R_g) \end{array}$$

Here, to understand why ι_ϵ makes sense, we need the following fact, the proof of which is left as an exercise.

Observation 5. $S_\delta(S_\epsilon(R_f)) = S_{\delta+\epsilon}(R_f)$.

Note that by construction of ι, ι_ϵ and φ_ϵ (or ψ_ϵ, respectively), the trapezoidal parts of this diagram commute trivially: $\varphi_\epsilon \circ \iota(x) = \varphi_\epsilon([x, 0]) = [\varphi(x), 0] = \iota_\epsilon \circ \varphi(x)$.

Furthermore, note that for sufficiently large ϵ, $S_\epsilon(R_f)$ is a union of segments, i.e., any two Reeb graphs of compact connected spaces are ϵ-interleaved for some ϵ.

Lastly, if R_f and R_g are ϵ-interleaved, then they are also δ-interleaved for all $\delta \geq \epsilon$.

Definition 6. $d_I(R_f, R_g) = \inf\{\epsilon \mid R_f, R_g$ *are* ϵ*-interleaved*$\}$.

Theorem 7. *For tame functions* $f, g : X \to \mathbb{R}$ *we have* $d_I(R_f, R_g) \leq \|f - g\|_\infty$.