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Introduction to Topological Data Analysis **Scribe Notes 23** **FS23**

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Multiscale Mapper

The multiscale Mapper is a combination of the ideas of persistence and of Mapper. We here want to look at different covers.

Definition 1. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{F} = \{V_\beta\}_{\beta \in B}$ be two covers of the same space X. A* map of covers *is a map $\varphi : A \to B$ such that for every $\alpha \in A$, we have $U_\alpha \subseteq V_{\varphi(\alpha)}$.*

Proposition 2. *If $\varphi : \mathcal{U} \to \mathcal{V}$ is a map of covers (with a slight abuse of notation), then the map $N(\varphi) : N(\mathcal{U}) \to N(\mathcal{V})$ given on the vertices by φ is simplicial.*

Proof. Let $\sigma \in N(\mathcal{U})$. We need to show that the intersection $\bigcap_{\beta \in \varphi(\sigma)} V_\beta$ is non-empty.

$$\bigcap_{\beta \in \varphi(\sigma)} V_\beta = \bigcap_{\alpha \in \sigma} V_{\varphi(\alpha)} \supseteq \bigcap_{\alpha \in \sigma} U_\alpha \neq \emptyset$$

Thus, $\varphi(\sigma) \in N(\mathcal{V})$. □

Proposition 3. *Let $f : X \to Z$ be some map, and $\mathcal{U}, \mathcal{V}$ be covers of Z, with $\varphi : \mathcal{U} \to \mathcal{V}$ some map of covers. Then, there exists a map of covers $f^*(\varphi) : f^*(\mathcal{U}) \to f^*(\mathcal{V})$.*

Recall that $f^*(\mathcal{U})$ is the cover of X consisting of the connected components of the preimages of the sets of $\mathcal{U}$ under f.

Proof. For every α, we have $U_\alpha \subseteq V_{\varphi(\alpha)} \implies f^{-1}(U_\alpha) \subseteq f^{-1}(V_{\varphi(\alpha)})$. We now need to go from these preimages to their connected components. Since every connected component of $f^{-1}(U_\alpha)$ must lie in a unique connected component of $f^{-1}(V_{\varphi(\alpha)})$, our desired map of covers is given by exactly mapping to this connected component. □

If we have multiple maps of covers, $\mathcal{U} \xrightarrow{\varphi} \mathcal{V} \xrightarrow{\psi} \mathcal{W}$, we can concatenate the maps, and the f^* function distributes: $f^*(\psi \circ \varphi) = f^*(\psi) \circ f^*(\varphi)$.

Let $\mathfrak{U} = \mathcal{U}_1 \xrightarrow{\varphi_1} \mathcal{U}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} \mathcal{U}_n$ be a sequence of covers of Z with maps between them, which we call a *cover tower*. By applying f^* we get a cover tower $f^*(\mathfrak{U})$ of X.

Definition 4 (Multiscale Mapper). *Let* $f : X \to Z$, $\mathfrak{U}$ *a cover tower of* Z. *Then, the* Multiscale Mapper $MM(\mathfrak{U}, f)$ *is*

$$MM(\mathfrak{U}, f) := N(f^*(\mathfrak{U})) = \{N(f^*(\mathcal{U}_i)) \mid \mathcal{U}_i \in \mathfrak{U}\})$$

together with the induced simplicial maps

$$N(f^*(\varphi_i)) : N(f^*(\mathcal{U}_i)) \to N(f^*(\rangle + \infty)).$$

Applying homology, we get the sequence homology groups with induced homomorphisms between them, i.e., a persistence module:

$$H_p(N(f^*(\mathcal{U}_1))) \overset{N(f^*(\varphi_1))}{\to} \dots \overset{N(f^*(\varphi_{n-1}))}{\to} H_p(N(f^*(\mathcal{U}_n))).$$

We can now view $\mathrm{Dgm}_p MM(\mathfrak{U}, f)$ *as a topological summary of* f *through the lens of* $\mathfrak{U}$.

Where already the normal Mapper has so many input parameters that using it is somewhat of an art form more than a single tool we can just throw onto a topological space, the Multiscale Mapper adds even more parameters. But, in some way a cover tower can also be seen as a way of looking at a whole interval of some of the parameters. For example, we can get a cover tower by increasing the size of all intervals in an interval cover. The features of the data should show up as a robust feature that persists for a longer time over this process, while spurious features obtained from choosing "wrong" Mapper parameters should disappear quickly.