Chapter 2

Mathematical Foundations

2.1 Topological Spaces

Topology, sometimes also called "rubber-sheet geometry", stems from the Greek words *tópos*, which means place or locality, and *lógos*, which means study. So, it can be roughly translated as the study of places and shapes. Indeed, as the name rubber-sheet geometry suggests, topology studies similar objects as geometry, but in a setting where properties are preserved under continuous deformations like stretching and twisting. In particular, these properties should be independent of metrics, but we would still like to have ways to describe proximity between points. We do this by looking at open neighborhoods of points. The core objects in topology are *topological spaces*, whose definition captures the system of open neighborhoods of the points in the space.

Definition 2.1. *A topological space* (X, T) *is a set of points* X*, with a system* T *of subsets of* X *(called the topology on* X*), such that*

1. $\emptyset \in T$, $X \in T$.
2. *For every* $S \subseteq T$, $\bigcup S \in T$.
3. *For every* **finite** $S \subseteq T$, $\bigcap S \in T$.

The sets in T *are called the* open *sets of* X.

For example, setting $X = \mathbb{R}^2$ and T to be the collection of open subsets (in the geometric/calculus sense) of $\mathbb{R}^2$, we can check that (X, T) is a topological space. A further example of a topological space is $(X, 2^X)$, where 2^X denotes the family of all subsets of X. This is called a *discrete topology.*

Another example is the Euclidean space $X = \mathbb{R}^d$, where the open sets T are defined as we know from calculus. This example also shows why we restrict the third condition of the definition above only to *finite* intersections of open sets: If we allowed infinite intersections in Condition 3, a set $\{p\}$ consisting of a single point $p \in \mathbb{R}^d$ (which by the calculus definition is not an open set) would have to be considered to be open; it is the intersection of the infinite series of open balls of radius $1/n$ centered at p, for $n \in \mathbb{N}$.

In most applications in these lecture notes, we work with subspaces of this Euclidean space $\mathbb{R}^d$. In that context we not only know the notion of open sets from calculus, but also notions such as *closed sets, closure, interior* and *boundary.* These terms can also be defined for abstract topological spaces:

Definition 2.2. *A set $Q \subseteq X$ is called* closed*, if its complement $X \setminus Q$ is open. The* closure $\operatorname{cl} Q$ *is the smallest closed set containing Q. The* interior $\operatorname{int} Q$ *is the union of all open subsets of Q. The* boundary $\operatorname{bnd} Q$ *is the set minus its interior:* $\operatorname{bnd} Q = Q \setminus \operatorname{int} Q$.

Note that sets can be open and closed simultaneously: in every topological space (X, T), $\emptyset$ and X are such examples. In a discrete topology, every subset $S \subseteq X$ is both open and closed.

Exercise 2.3. *Show that a finite union of closed sets is closed.*

So far we have only seen two topological spaces: Euclidean space, and the (rather boring) discrete topology on any set X. In order to see the value in the abstractions we are doing, we would like to have more examples of topological spaces. In particular, it would be great if we had a way to get new topological spaces from known ones. In the following we discuss some ways to do this, starting with taking intersections.

Lemma 2.4. *Let (X, T) be some topological space, and $Y \subseteq X$. Then, $U := \{A \cap Y \mid A \in T\}$ is a topology on Y. We call this a* subspace topology.

Proof. We check the three conditions of a topology:

1. $\emptyset = \emptyset \cap Y$, therefore $\emptyset \in U$. Similarly, $Y = X \cap Y$, and thus $Y \in U$.
2. $\bigcup_{i \in I}(A_i \cap Y) = (\bigcup_{i \in I} A_i) \cap Y$, and thus $\bigcup_{i \in I}(A_i \cap Y) \in U$.
3. $\bigcap_{i=1}^{n}(A_i \cap Y) = (\bigcap_{i=1}^{n} A_i) \cap Y$, and thus $\bigcap_{i=1}^{n}(A_i \cap Y) \in U$. □

Since we have seen a natural topology on $\mathbb{R}^d$, this already gives us a natural topology for all subsets of $\mathbb{R}^d$.

Another way to get topological spaces is as a product of spaces. We will not discuss the details of this here, and refer the interested reader to any textbook on topology, such as the excellent book by Munkres [2].

Fact 2.5. *Let $(X, T_X), (Y, T_Y)$ be two topological spaces. Then there exists a topology on $X \times Y$, called the* product topology.

Finally, we can also get a topological space by taking the union of two disjoint topological spaces. If a space can be obtained as such a union, we call it disconnected:

Definition 2.6. *A topological space* (X, T) *is* disconnected*, if there are two disjoint non-empty open sets* $U, V \in T$*, such that* $X = U \cup V$*. A topological space is* connected*, if it is not disconnected.*

Exercise 2.7. *In this exercise, we will use topology to prove that the set of primes is infinite.*

We define the sets $S(a, b)$ *as follows:*

$$S(a, b) := \{an + b \mid n \in \mathbb{Z}\}, \ \forall a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z}$$

We then say that a set $U \subseteq \mathbb{Z}$ *is open, if and only if for all* $x \in U$*, there exists* $a \in \mathbb{Z}$ *such that* $S(a, x) \subseteq U$*. This is equivalent to saying that every open set* U *is a union of zero or more (including infinitely many) sets* $S(a, b)$*.*

(a) Show that this defines a topology on $\mathbb{Z}$*.*

(b) Let $A \subset \mathbb{Z}$ *be finite and non-empty. Show that* $\mathbb{Z} \setminus A$ *cannot be closed.*

(c) Show that $S(a, b)$ *is both open and closed.*

(d) Show that

$$\bigcup_{p \text{ prime}} S(p, 0) = \mathbb{Z} \setminus \{-1, 1\}$$

(e) Conclude that there are infinitely many primes.

2.2 Metric Spaces

Recall that topological spaces should capture neighborhoods of points without requiring the notion of a distance. However, if we do have distances, we should still be able to use the framework of topological spaces. In other words, topological spaces should be a generalization of spaces with distances.

Definition 2.8. *A* metric space (X, d) *is a set* X *of points and a distance function* $d : X \times X \to \mathbb{R}$ *satisfying*

1. $d(p, q) = 0$ *if and only if* $p = q$.
2. $d(p, q) = d(q, p)$, $\forall p, q \in X$. *(Symmetry)*
3. $d(p, q) \leqslant d(p, s) + d(s, q)$, $\forall p, q, s \in X$. *(Triangle inequality)*

Note that these three conditions imply that $d(p, q) \geqslant 0$ for all $p, q \in X$: If some distance $d(p, q)$ would be negative, we would have $0 = d(p, p) \leqslant d(p, q) + d(q, p) = 2 \cdot d(p, q) < 0$, a contradiction.

Fact 2.9. *Every metric space has a topology (the* metric space topology*) given by the open metric balls* $B(c, r) = \{p \in X \mid d(p, c) < r\}$ *and their unions.*

2.3 Maps Between Topological Spaces

In most areas of mathematics, there are two things that are at the core of every theory: objects, and mappings between them. For example, in linear algebra we study vector spaces and the linear maps between then. Now that we have defined the objects of study — topological spaces — we want to look at the mappings between them.

Definition 2.10. *A function* $f : X \to Y$ *is* continuous *if for every open set* $U \subseteq Y$*, its pre-image* $f^{-1}(U) \subseteq X$ *(the set of all elements* $x \in X$ *such that* $f(x) \in U$*) is open. Continuous functions are also called* maps*. If* f *is an injective map, it is called an* embedding*.*

Let us give some examples:

- For $X \subseteq Y$, we write $X \hookrightarrow Y$ for the function $f(x) = x, \forall x \in X$. This function, which is also called the *inclusion map*, is continuous: $f^{-1}(U) = U \cap X$, which is open in the subspace topology on X.
- For a function $f : \mathbb{R} \to \mathbb{R}$, continuity agrees with the "ϵ-δ" definition of continuity from calculus.

Exercise 2.11. *A topological space* (X, T) *is called* path-connected *if any two points* $x, y \in X$ *can be joined by a path, i.e., there exists a map* $f : [0, 1] \to X$ *of the segment* $[0, 1] \subset \mathbb{R}$ *onto* X *such that* $f(0) = x$ *and* $f(1) = y$*. Prove that a path-connected space is connected.*

An important question we have to answer is when we want to consider two topological spaces to be "the same". In the rest of this section we develop some notions of equivalence of topological spaces, each based on the existence of some continuous function(s).

Definition 2.12. *A* homeomorphism *is a bijective map* $f : X \to Y$ *whose inverse is also continuous. Two topological spaces are* homeomorphic*, if there is a homeomorphism between them. We also write* $X \simeq Y$ *to say that* X, Y *are homeomorphic.*

To make sure that homeomorphism is a reasonable notion of equivalence, we should check that it is indeed an equivalence relation.

Exercise 2.13. *Show that the relation of being homeomorphic is an equivalence relation, that is, show that every space is homeomorphic to itself, show that the relation is symmetric (*$X \simeq Y$ *iff* $Y \simeq X$*), and show that* $\simeq$ *is transitive (if* $X \simeq Y$ *and* $Y \simeq Z$*, then* $X \simeq Z$*).*

Let us apply our definition to some examples, to see whether it captures our intuition:

- The boundary of a tetrahedron is homeomorphic to the sphere S^2 (with both spaces considered as a subspace of $\mathbb{R}^3$). A homeomorphism can be found by taking a point c in the interior of the tetrahedron, and sending each point p of the boundary to the point $f(p)$ on the ray from c through p such that $d(c, f(p)) = 1$.

- The open interval $I := (-1, 1)$ is homeomorphic to $\mathbb{R}$. The following map f is a homeomorphism: $f : I \to \mathbb{R}$, $x \mapsto \frac{x}{1-|x|}$. Its inverse is $f^{-1} : \mathbb{R} \to I$, $y \mapsto \frac{y}{1+|y|}$.
- All knots (a knot is the image of an embedding of the circle into $\mathbb{R}^3$) are homeomorphic. Thus, we cannot distinguish between knots using only homeomorphism.

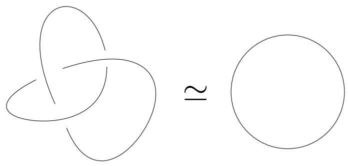
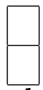


Figure 2.1: *Two knots.*

Exercise 2.14. *Give an example of a map $f : X \to Y$ that is bijective but not a homeomorphism.*

Exercise 2.15. *Consider a grid of 2 vertical line segments and $k + 2$ horizontal segments, for some $k \geqslant 0$. For $k = 1$, this looks as follows:* 
Now, we consider the problem of placing a point on each of the $k + 2$ horizontal line segments, such that each of the $k + 4$ total line segments contains at least one point.

(a) How could one define a topology on the set of all such point placements?

(b) Convince yourself that this space is homeomorphic to S^k.

The example of the knots shows that in certain cases, homeomorphism does not capture all the information we can use to distinguish two spaces. In this example, this distinguishing information is not really stored in the topological spaces themselves, but in the way they are embedded in the "ambient" space (in this example $\mathbb{R}^3$). In such a case, where the two spaces we consider are both embedded into the same ambient space, we can not only look at maps between the two spaces themselves, but we can also consider whether one of them can be continuously deformed into the other:

Definition 2.16 ([1, Def. 1.18]). *An* isotopy *connecting $X \subseteq A$ and $Y \subseteq A$ is a continuous map $\phi : X \times [0, 1] \to A$, such that $\phi(X, 0) = X$, $\phi(X, 1) = Y$, and $\forall t \in [0, 1]$, $\phi(\cdot, t)$ is a homeomorphism between X and its image. Two spaces are called* isotopic, *if there is an isotopy connecting them.*

Exercise 2.17. *Show that the relation of being isotopic is an equivalence relation.*

Let us check isotopy on a few examples, starting with the knots from above:

- The two knots from Figure 2.1 above (embedded in $A = \mathbb{R}^3$) are homeomorphic but not isotopic. Isotopy thus captures our intuition more accurately than homeomorphism in this case.
- Let $X \subset \mathbb{R}$ be the union of $\{0\}$, and $[1, 2]$, and let $Y \subset \mathbb{R}$ be the union of $[0, 1]$ and $\{2\}$. These spaces are homeomorphic ($X \simeq Y$), but not isotopic. Just as with the knots, the difference between these spaces does not lie in their topology, but in the way they are embedded into the ambient space $\mathbb{R}$.
- Consider the two spaces in Figure 2.2, which are considered to be embedded in the ambient space A consisting of $\mathbb{R}^3$ minus the grey infinitely long pole in the middle. Do you think the spaces are isotopic? Most people would probably argue that they are not, as in the left space both loops of the handcuff are locked around pole while in the right space one loop is free. However, it turns out that the spaces are in fact isotopic. An isotopy is illustrated by the following video: https://www.youtube.com/watch?v=wDZx9B4TAXo

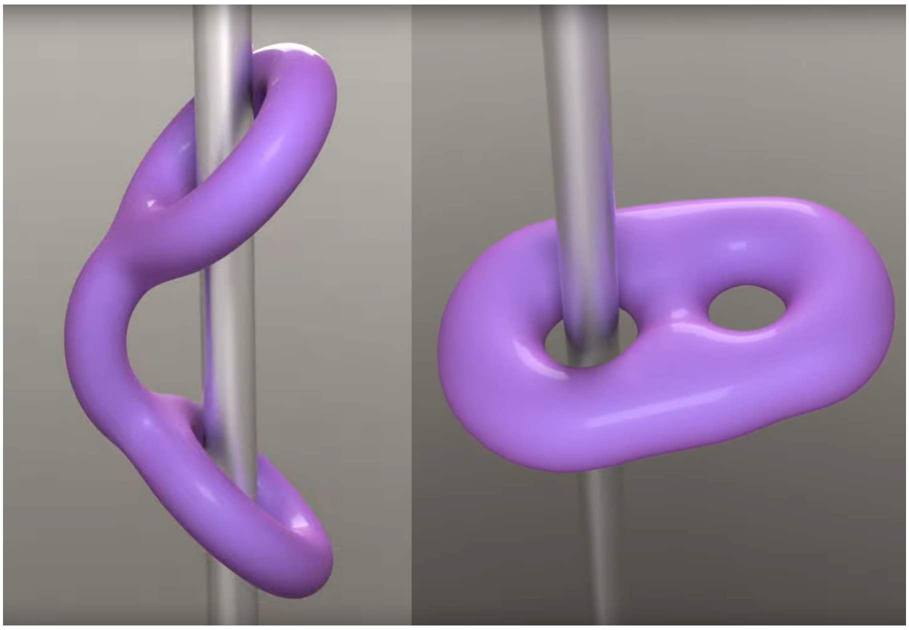


Figure 2.2: *Left: Both loops of the handcuffs are wrapped around an infinite pole. Right: Only one loop of the handcuffs is wrapped around the infinite pole. These spaces are isotopic.*

Using isotopy we have now managed to distinguish between two spaces (embeddings) that homeomorphism could not distinguish. On the other hand, homeomorphism is also very restrictive: For example, any two-dimensional space X (such as the mantle of a cylinder) cannot be homeomorphic to any one-dimensional space Y (such as a circle), simply due to the difference in cardinality of X and Y. We thus also want to develop a weaker notion of equivalence than homeomorphism.

To do this, we take the idea of continuous deformations from isotopy, but instead of applying it to deform spaces into each other, we deform maps into each other:

Definition 2.18. *Let* g, h *be maps* $X \to Y$. *A* homotopy *connecting* g *and* h *is a map* $H : X \times [0, 1] \to Y$ *such that* $H(\cdot, 0) = g$ *and* $H(\cdot, 1) = h$. *In this case* g *and* h *are called* homotopic.

Before we use homotopies to define an equivalence on topological spaces, let us again consider some examples:

- The inclusion map $g : \mathbb{B}^3 \hookrightarrow \mathbb{R}^3$ (where $\mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$), and the constant map $h : \mathbb{B}^3 \to \mathbb{R}^3$ which sends every point to the origin, are homotopic, as shown by the homotopy
$$H(x, t) = (1 - t)g(x).$$
- The identity map $g : S^1 \to S^1$, and the constant map $h : S^1 \to S^1$ which sends everything to a single point $p \in S^1$, are *not* homotopic.

The notion of homotopy now allows us to define our desired equivalence relation on topological spaces that is weaker than homeomorphism. Intuitively, this relation says that two spaces are the same if they can be continuously transformed into each other not only by bending, twisting and stretching, but also by shrinking or blowing up parts of different dimensions. However, note that unlike with isotopy, we do not need to consider the two spaces to be embedded in any ambient space.

Definition 2.19. *Two spaces* X, Y *are* homotopy equivalent *if there exist maps* $g : X \to Y$ *and* $h : Y \to X$ *such that:*

- $h \circ g$ *is homotopic to* id_X *(the identity map* $x \mapsto x$*), and*
- $g \circ h$ *is homotopic to* id_Y.

Exercise 2.20. *Show that the relation of being homotopy equivalent is an equivalence relation.*

Let us consider some examples:

- The circle S^1 and $\mathbb{R}^2 \setminus \{0\}$ are homotopy equivalent: We pick g as the inclusion map $S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$, and $h(x) := \frac{x}{|x|}$. We see that $h \circ g(x) = x$, i.e., $h \circ g = \mathrm{id}_{S^1}$. Furthermore, $g \circ h(x) = h(x)$. Finally, $g \circ h$ and $\mathrm{id}_{\mathbb{R}^2 \setminus \{0\}}$ are homotopic as certified by the homotopy $H(x, t) := tx + (1 - t)h(x)$.
- The cylinder mantle and the circle are homotopy equivalent, but not homeomorphic.
- Any ball $\mathbb{B}^d$ is homotopy equivalent to the single point. We call such spaces *contractible*.

The next lemma shows that homotopy equivalence is a strictly weaker notion than homeomorphism:

Lemma 2.21. *If X and Y are homeomorphic, they are also homotopy equivalent.*

Proof. Let $g : X \to Y$ be the homeomorphism, and $h := g^{-1}$ its inverse. Then $g \circ h = id_Y$ and $h \circ g = id_X$, and id is homotopic to itself. □

With the need for two maps and a proof that they are homotopic, proving homotopy equivalence directly can be quite tedious. The following notion of *deformation retracts* gives an easy way of proving homotopy equivalence in some cases.

Definition 2.22. *Let $A \subseteq X$. A* deformation retract *of X onto A is a map $R : X \times [0, 1] \to X$, such that*

- $R(\cdot, 0) = id_X$
- $R(x, 1) \in A, \forall x \in X$
- $R(a, t) = a, \forall a \in A, t \in [0, 1]$

If such a deformation retract of X onto A exists, we also say that A is *a deformation retract of X.*

The intuition behind a deformation retract is that the map R continuously shrinks X to A, while leaving A fixed. Note that unlike homeomorphism, isotopy and homotopy equivalence, deformation retracts are inherently asymmetric.

Fact 2.23. *If A is a deformation retract of X, then A and X are homotopy equivalent.*

Let us use this to prove homotopy equivalence of some examples:

- The circle S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$: $R(x, t) = (1 - t)x + t \cdot \frac{x}{|x|}$. Note how much easier this is to prove without needing to use the two maps h and g as above.
- A punctured torus can be deformation retracted onto the symbol 8 where one of the two circles is rotated by 90°, as seen by the following video: https://www.youtube.com/watch?v=tz3QWrfPQj4

One may think that deformation retracts are only useful for proving homotopy equivalence when one space is a subspace of the other. However, the following fact shows that deformation retracts can prove homotopy equivalence of any pair of spaces:

Fact 2.24. *X, Y are homotopy equivalent if and only if there exists a space Z such that X and Y are deformation retracts of Z.*

An example of this fact can be found in Figure 2.3.

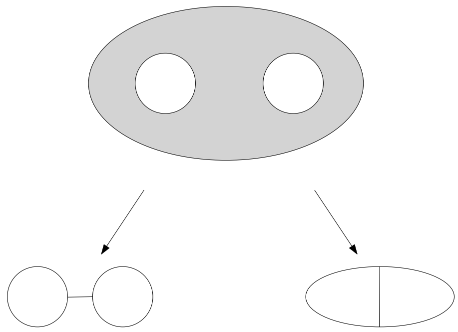
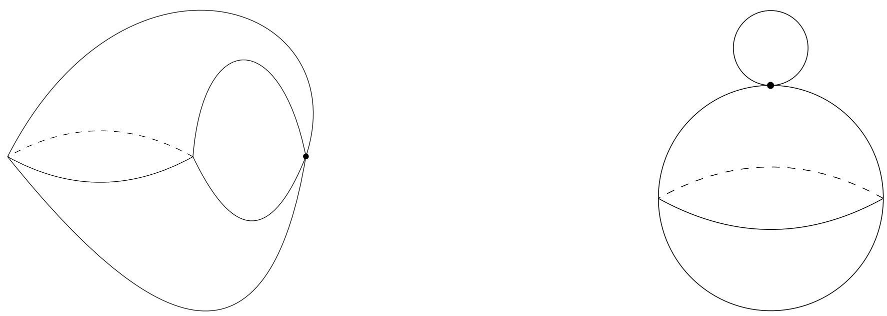


Figure 2.3: *The top space deformation retracts to both spaces below, showing that they are homotopy equivalent.*

Exercise 2.25. *Sort the letters of the alphabet into equivalence classes under homotopy equivalence.*

Exercise 2.26. *Show that both a cylinder and a Möbius strip are homotopy equivalent to a circle.*

Exercise 2.27. *Let* X *be* S^2 *where the north pole and the south pole have been glued together, see Figure 2.4a. Let* Y *be* S^2 *with an* S^1 *attached at the north pole, see Figure 2.4b.*



(a) *The space* X*.* **(b)** *The space* Y*.*

Figure 2.4: *The spaces from Exercise 2.27.*

Give an informal argument that X *and* Y *are homotopy equivalent. Bonus question: Are they also homeomorphic?*

We note that in general showing existence of a map with certain properties (e.g., a homeomorphism, isotopy, homotopy) is easy: just give a map and show that it satisfies the required properties. On the other hand, showing that such a map cannot exist is hard, as there are usually infinitely many candidate maps. The idea of algebraic topology is to identify invariant properties preserved by these maps. Then, we know that no map can exist between spaces on which these invariants differ. An example of such an invariant is the number of "holes" a space has, which we will formalize when we introduce the notion of *homology*.

2.4 Algebra

In this section we recap the necessary background in algebra that is needed for the basics of homology theory. Just as in the previous sections, we first introduce the objects of study, followed by the maps between them.

Definition 2.28. *A group* $(G, +)$ *is a set* G *together with a binary operation "+" such that*

1. $\forall a, b \in G$: $a + b \in G$
2. $\forall a, b, c \in G$: $(a + b) + c = a + (b + c)$ *(Associativity)*
3. $\exists 0 \in G$: $a + 0 = 0 + a = a \; \forall a \in G$
4. $\forall a \in G \exists - a \in G$: $a + (-a) = 0$

$(G, +)$ *is* abelian[1] *if we also have*

5. $\forall a, b \in G$: $a + b = b + a$ *(Commutativity)*

Let us point out some examples:

- $(\mathbb{Z}, +)$ is a group (even an abelian one), but $(\mathbb{N}, +)$ is not, since any non-zero number does not have an inverse element.
- Consider the (very large) set of all sequences of moves of a Rubik's cube that do not contain a subsequence equivalent to doing nothing. This set forms a group (with the "+" operation being concatenation), but not an abelian one: let L denote moving the left face clockwise, and let U denote moving the upper face clockwise. Replacing "clockwise" by counter-clockwise we get $-L$ and $-U$, respectively. Now, if the group was abelian, then $L+U-L-U$ should give the same configuration again, but if you do these moves on a Rubik's cube, you will see that the configuration has changed.

As groups can be very large, even infinitely large, it can be useful to have a concise way of writing them:

Definition 2.29. *Let* $(G, +)$ *be a group.*
A subset $A \subseteq G$ *is a* generator *if every element of* G *can be written as a finite sum of elements of* A *and their inverses.*
A subset $B \subseteq G$ *is a* basis *if every element of* G *can be* uniquely *written as a finite sum of elements of* B *and their inverses (ignoring trivial cancellations, i.e.,* $a + c + (-c) + (-b) = a + (-b)$*).*
An abelian group that has a basis is called free*.*

[1]Note that unlike other mathematical concepts named after a person, *abelian* is usually not capitalized.

Examples:

- The six standard moves of the Rubik's cube (rotating the top, bottom, front, back, left, or right layer clockwise by $90°$) are a generator for the Rubik's cube move sequences.
- $\{1\}$ is a basis of $(\mathbb{Z}, +)$.

Exercise 2.30. *A cyclic group is a group* G *that contains an element* $g \in G$ *such that* $\{g\}$ *is a generator of* G*. Show that every cyclic group is abelian (commutative).*

Exercise 2.31. *Consider a Rubik's cube. Prove that no sequence* X *of elementary moves exists such that every Rubik's cube can be solved by repeatedly applying* X*.*

Definition 2.32. *For some group* $(G, +)$*,* $H \subseteq G$ *is a* subgroup*, if* $(H, +)$ *is also a group.*

For example, the even integers (including 0) are a subgroup of $(\mathbb{Z}, +)$. Subgroups are important in group theory, as they can be used to partition a group into several parts:

Definition 2.33. *Let* $H \subseteq G$ *be a subgroup of* $(G, +)$*, and* $a \in G$*.*
The left coset $a + H$ *is the set* $a + H := \{a + b \mid b \in H\}$*, and the* right coset $H + a := \{b + a \mid b \in H\}$*. If* G *is abelian,* $a + H = H + a$*, and they are simply called the* coset*.*
For G *abelian, the* quotient group of G by H*, denoted by* G/H*, is the group on the set of cosets* $\{a+H, a \in G\}$ *with the operation* $\oplus$ *defined as* $(a+H) \oplus (b+H) = (a+b)+H$*,* $\forall a, b \in G$*.*

Examples:

- Let $G = (\mathbb{Z}, +)$ and $H = n\mathbb{Z} = \{n \cdot a \mid a \in \mathbb{Z}\}$. Then, $G/H = \{0 + \mathbb{Z}, 1 + \mathbb{Z}, \ldots, (n-1) + \mathbb{Z}\}$ is the group usually referred to as $\mathbb{Z}_n$, the group of modular arithmetic modulo n.
- $\mathbb{R}/\mathbb{Z}$ is the circle group (the multiplicative group of all complex numbers of absolute value 1). Try and convince yourself of this!

In order to compare groups with each other, we again want a notion of maps between groups, that behave well with the group structures:

Definition 2.34. *A map* $h : G \to H$ *between abelian groups* $(G, +)$ *and* $(H, \star)$ *is a* homomorphism *if* $h(a + b) = h(a) \star h(b)$*,* $\forall a, b \in G$*.*
A bijective homomorphism is called an isomorphism*, and then we write* $G \cong H$ *and say that* G *and* H *are* isomorphic*.*

kernel $\ker h := \{a \in G \mid h(a) = 0\}$

image $\operatorname{im} h := \{b \in H \mid \exists a \in G \text{ with } h(a) = b\}$

cokernel $\operatorname{coker} h := H/\operatorname{im} h$

Note that we are assuming something in our definition of the cokernel: for the definition of a quotient group to apply, we need the divisor group to be a subgroup of the dividend group. Luckily, the following lemma says that $\operatorname{im} h$ is always a subgroup of H.

Lemma 2.35. $\ker h$ *and* $\operatorname{im} h$ *are subgroups of* $(G, +)$ *and* $(H, \star)$*, respectively.*

Proof. We first prove this for $\ker h$.

1. $a, b \in \ker h \Rightarrow h(a) = h(b) = 0$. By definition of homomorphism, $h(a + b) = h(a) \star h(b) = 0 \star 0 = 0$, and thus by definition of $\ker h$, $a + b \in \ker h$. We conclude that $\ker h$ is closed under $+$.

2. Associativity follows from associativity of $+$ in G, since $\ker h \subseteq G$.

3. $\forall a \in G:\ h(0) \star h(a) = h(0 + a) = h(a)$, and thus $h(0) = 0$, from which $0 \in \ker h$ follows.

4. Let $a \in \ker h$. Then, $0 = h(0) = h(a - a) = h(a) \star h(-a) = 0 \star h(-a) = h(-a)$, and thus $-a \in \ker h$.

The proof for $\operatorname{im} h$ is left as an exercise. □

Exercise 2.36. *Show that* $\operatorname{im} h$ *is a subgroup of* H.

Exercise 2.37. *For two abelian groups* $(G, \star)$ *and* $(H, +)$*, let the set of all homomorphisms* $f : G \to H$ *be denoted by* $\operatorname{Hom}(G, H)$.

(a) Show that $(\operatorname{Hom}(G, H), \oplus)$*, where the operation* $\oplus$ *is defined as*

$$(f \oplus g)(x) = f(x) + g(x), \forall x \in G,$$

is also a group.

(b) Show that $\operatorname{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2) \cong \mathbb{Z}_2^2$.

As the example of the integers shows, a big motivation for the study of groups comes from number theory. However, in number theory we do not only have addition but also multiplication. This motivates the following definition:

Definition 2.38. $(R, +, \cdot)$ *is a* ring*, if*

1. $(R, +)$ *is an abelian group.*

2. $\forall a, b, c \in R$*:*
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *and* *(Associativity of* $\cdot$*)*
$a \cdot (b + c) = a \cdot b + a \cdot c,$
$(b + c) \cdot a = b \cdot a + c \cdot a$ *(Distributivity)*

3. *$\exists 1 \in R$, such that $a \cdot 1 = 1 \cdot a = a \; \forall a \in R$.* *(Multiplicative identity)*

If $\cdot$ is commutative, we say that R is commutative.

Definition 2.39. *A commutative ring in which every non-zero element has a multiplicative inverse (i.e., $\forall a \in R \setminus \{0\}, \exists b \in R : a \cdot b = 1$) is called a* field.

Another important area of algebra, which you already know, is linear algebra. Here, vectors can be added and subtracted. Further the field of real numbers are called *scalars* and they can be multiplied with vectors. So, we have very similar operations at hand. This motivates the following generalization of the concept of vector spaces.

Definition 2.40. *Given a ring $(R, +, \cdot)$ with multiplicative identity 1, an* R-module M *is an abelian group $(M, \oplus)$ with an operation $\otimes : R \times M \to M$ such that for all $r, r' \in R$ and $x, y \in M$, we have*

1. $r \otimes (x \oplus y) = (r \otimes x) \oplus (r \otimes y)$
2. $(r + r') \otimes x = (r \otimes x) \oplus (r' \otimes x)$
3. $1 \otimes x = x$
4. $(r \cdot r') \otimes x = r \otimes (r' \otimes x)$

If R is a field, the R-module is called a vector space.

In the literature, often the same symbol $(\cdot)$ is used for both operations $\cdot$ and $\otimes$, and $+$ for both $+$ in R and $\oplus$ in M. For a vector space, this should feel quite normal, since for the vector space $\mathbb{R}^n$ (which is an $\mathbb{R}$-module), we also write $\cdot$ for multiplying scalars to both scalars and vectors, and $+$ for addition of both scalars and vectors.

Modules appear all over the place in homology theory. In some cases, in particular in all the cases we discuss in these lecture notes, the modules happen to be vector spaces. Thus, most of what we discuss in the following chapters could be phrased using only language from linear algebra. However, to be consistent with most of the existing literature, we will phrase most results in a slightly more general language.

Questions

1. *What is a topological space?* Give the formal definition and some examples.
2. *What is a continuous map between topological spaces? What is a homeomorphism?* State the definitions and give examples.
3. *What is a homotopy? What is a homotopy equivalence?* Give the formal definitions. Further, define deformation retracts and use them to give an alternative definition of homotopy equivalence.
4. *What are groups and the maps between them?* State the definitions and prove that the image and kernel are subgroups.

References

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[2] J.R. Munkres, *Topology*, Prentice Hall, Incorporated, 2000.