Chapter 3

Homology

In this chapter, we introduce *homology*, a fundamental concept in algebraic topology and, as the name suggests, a crucial element of the *persistent homology pipeline* in topological data analysis. Very informally, homology can be used to count the number of "holes" of a topological space, where holes can have any dimension. While you might have an intuition of what a 2-dimensional hole in a subspace of $\mathbb{R}^2$ might be, it is not at all clear what a 4-dimensional hole in some 7-dimensional space should be. The main idea of homology is to use algebra to talk about holes in an abstract setting.

As we have already hinted at in the previous chapter, homology is an invariant of topological spaces preserved under homeomorphism and homotopy equivalence. We will manage to make this formal in Section 3.2.8.

3.1 Simplicial Complexes

In order to define homology, we restrict our attention (for now) to special types of topological spaces, namely *simplicial complexes*. We will see that this covers most natural spaces. Furthermore, homology for simplicial complexes is sufficient for all classical applications in topological data analysis. We will briefly outline a more general definition later in the chapter.

While simplicial complexes can be regarded as completely abstract objects, it is more intuitive to think of them in a geometric setting. The basic objects in a geometric simplicial complex are *simplices*:

Definition 3.1. *A* k-simplex *in* $\mathbb{R}^d$ *is the convex hull of* $k+1$ *affinely independent points in* $\mathbb{R}^d$*.*

A *face* of a simplex is the convex hull of a subset of its vertices. In particular, every face of a simplex is also a simplex. The empty set $\emptyset$ is also a face. The $(k-1)$-faces of a k-simplex are called *facets*. We say the *dimension* of a k-simplex is k.

Definition 3.2. *A* geometric simplicial complex *is a family* K *of simplices such that*

- *if* $\tau \in K$ *and* σ *is a face of* τ*, then* $\sigma \in K$*, and*
- *for* $\sigma, \tau \in K$*, their intersection* $\sigma \cap \tau$ *is a face of both.*

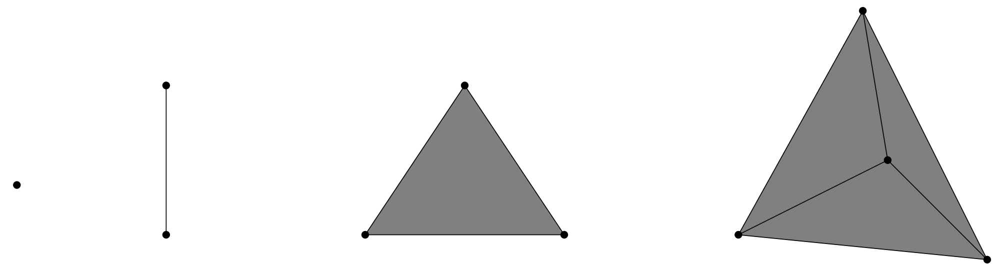


Figure 3.1: *Some examples of simplices: a point (0-dimensional), a line segment (1-dimensional), a triangle (2-dimensional) and a (filled) tetrahedron (3-dimensional).*

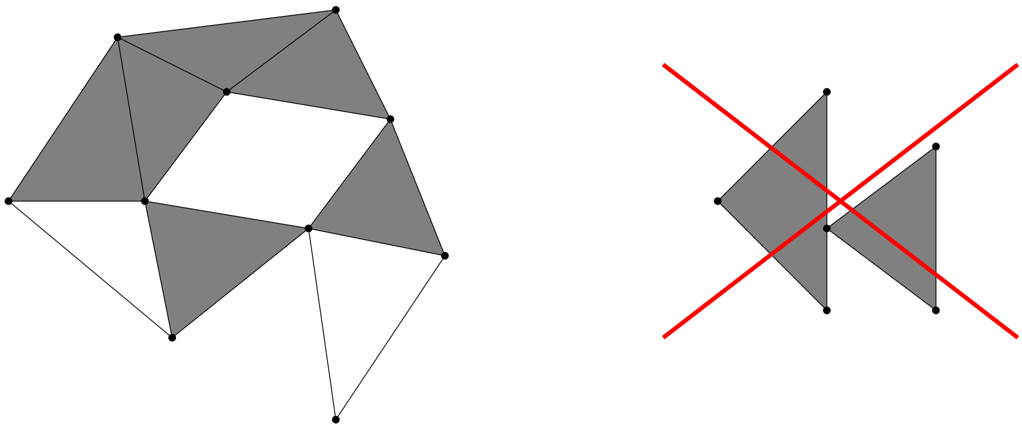


Figure 3.2: *The left is a simplicial complex. The right is not, as the intersection of the two triangles is not a face of both of them.*

We say the *dimension* of a simplicial complex is the maximum dimension of any simplex. In these lecture notes, and for applications in topological data analysis in general, we may assume that all simplicial complexes are *finite*, that is, consisting of finitely many simplices.

The way we defined them, simplicial complexes are geometric objects. To arrive at a purely combinatorial description, we can simply forget about the points in space spanning our simplices.

Definition 3.3. *An* abstract simplicial complex K *is a family of subsets of a vertex set* $V(K)$ *such that if* $\tau \in K$ *and* $\sigma \subseteq \tau$*, then* $\sigma \in K$*.*

A k-simplex here is a subset of $k+1$ elements, and thus again called k-dimensional. Note that 1-dimensional abstract simplicial complexes are exactly graphs: they are defined by a vertex set V and a system of two-element subsets of V, called *edges*.

From every geometric simplicial complex we get an abstract simplicial complex by simply taking the set of points as the vertex set and adding the correct subset for every simplex. For the inverse direction, we have to talk about geometric realizations:

Definition 3.4. *A geometric simplicial complex* K *is a* geometric realization *of some abstract simplicial complex* K'*, if there is an embedding* $e : V(K') \to \mathbb{R}^d$ *that takes every (abstract)* k*-simplex* $\{v_0, \dots, v_k\}$ *in* K' *to the (geometric)* k*-simplex that is the convex hull of* $e(v_0), \dots, e(v_k)$*.*

Does every abstract simplicial complex have a geometric realization? Let us only consider 1-dimensional complexes (graphs) for now. We know that not all graphs admit a straight-line embedding in the plane, as only planar graphs admit any embedding, i.e., crossing-free drawing, in the plane. However, by placing the vertices in $\mathbb{R}^3$ in such a way that no four vertices lie on a common plane, we see that we can always find a geometric realization of a graph in $\mathbb{R}^3$. This generalizes to the following realization theorem:

Theorem 3.5. *Every* k*-dimensional simplicial complex has a geometric realization in* $\mathbb{R}^{2k+1}$*.*

Proof. Place the vertices as distinct points on the *moment curve* in $\mathbb{R}^{2k+1}$, which is the curve given by $f(t) = (t, t^2, \dots, t^{2k+1})$. This way, any $2k+2$ of the placed points are affinely independent. Thus, any two faces with disjoint vertex sets will not intersect in the realization, showing that the realization is indeed an embedding. □

Since we now know that abstract and geometric simplicial complexes can be translated into one another, we will not make the distinction between them again and just use the word *simplicial complex* for both objects in the following. As a subset of Euclidean space, a simplicial complex thus also inherits the subspace topology from $\mathbb{R}^d$, which allows us to view simplicial complexes as topological spaces. We usually write K for the simplicial complex as a family of sets, and $|K|$ for the underlying topological space.

On the other hand, most topological spaces are not simplicial complexes. For example, the 2-sphere S^2 is not a simplicial complex, as it is not defined by a vertex set and faces. However, the boundary of a tetrahedron is a simplicial complex, and it is homeomorphic to S^2. Considering that we want to consider properties invariant under homeomorphism, we thus might as well work with the boundary of a tetrahedron instead of with S^2. This motivates the following definition.

Definition 3.6. *A simplicial complex* K *is a* triangulation *of a topological space* X*, if* $|K|$ *is homeomorphic to* X*. We say that a topological space* X *is* triangulable *if it has a triangulation.*

Triangulable spaces are nice for us, as we can replace them by simplicial complexes without any loss of topological information. Unfortunately, not all topological spaces are triangulable, but in this course we will not deal with such spaces.

While triangulations give us simplicial complexes from (triangulable) topological spaces, we would like to mention that one can also go the other way: many combinatorial structures naturally give rise to (abstract) simplicial complexes, which can in turn be interpreted as topological spaces. Thus, we can use the machinery of topological methods for gaining insights into many combinatorial problems. This gives rise to a subfield of combinatorics called *topological combinatorics*, where the topology of simplicial complexes associated to combinatorial objects is studied. Let us give some examples of such simplicial complexes.

- As we have already discussed, graphs are equivalent to 1-dimensional simplicial complexes.
- Given a graph $G = (V, E)$, we can define a simplicial complex on V by including a face $\{v_1, \dots, v_k\}$ whenever these vertices form a clique in G. This is called the *clique complex* of G.
- For a poset $(P, \leqslant)$, the set of all chains of P forms a simplicial complex, giving rise to the *order topology*.

In topological data analysis, a highly relevant example is the *nerve*, which records the intersection pattern of a collection of sets:

Definition 3.7. *For a finite collection* U *of sets, its* nerve $N(U)$ *is a simplicial complex on the vertex set* U *that contains* $u_0, \dots, u_k$ *as a* k*-simplex iff* $u_0 \cap \dots \cap u_k \neq \emptyset$.

While the nerve can be seen as a purely combinatorial object describing the intersection pattern of U, it is also interesting to study its topology. If the considered sets in U are subsets of some topological space X, there is a very strong characterization of the topology of $N(U)$, if the intersections of sets in U are "well-behaved".

Definition 3.8. *Let* X *be a topological space, and* U *a finite family of closed subsets of* X*. We call* U *a* good cover*, if every non-empty intersection of sets in* U *is contractible.*

Under these conditions on the sets we get the following, very powerful theorem, which allows us to relate complex spaces (unions of sets) with a much simpler simplicial complex, namely the nerve of these sets. For a proof of this we refer to any textbook on algebraic topology, for example the one by Hatcher [2].

Theorem 3.9 (Nerve theorem). *If* U *is a good cover, then* $|N(U)|$ *is homotopy equivalent to* $\bigcup U$.

The nerve theorem also holds if all the sets in U are open with contractible intersections, but it may fail if some sets in U are closed, and some open: We can have an open and a closed set which do not intersect, but whose union is connected.

Now that we have defined simplicial complexes and considered some examples, we once again want to study maps between them. The study of simplicial complexes and the maps between them, as we will define them, is called *combinatorial topology*.

Definition 3.10. *A* vertex map $f : V(K_1) \to V(K_2)$ *maps vertices in* K_1 *to vertices in* K_2*.*

Definition 3.11. *A map* $f : K_1 \to K_2$ *is called* simplicial *if it can be described by a vertex map* $g : V(K_1) \to V(K_2)$ *such that for every simplex* $\{v_0, \dots, v_k\}$ *we have* $f(\{v_0, \dots, v_k\}) = \{g(v_0), \dots, g(v_k)\}$*. Since* f *maps to* K_2 *we must have that* $f(\{v_0, \dots, v_k\})$ *is a simplex in* K_2*. A simplicial map can also be seen as a map on the underlying spaces* $f : |K_1| \to |K_2|$*.*

Note that for a map to be simplicial, we do not require that $\{f(v_0), \dots, f(v_k)\}$ is also a k-dimensional simplex, we merely require that it is *a* simplex of K_2. It is thus possible that distinct vertices of K_1 are mapped to the same vertex of K_2.

Recall that simplicial complexes are topological spaces, so there is also the notion of continuous maps between them. It can be shown that every simplicial map is continuous.

Exercise 3.12. *Let* $f : |K_1| \to |K_2|$ *be a simplicial map. Show that* f *is continuous.*

On the other hand, continuous maps in general do not need to map vertices to vertices, and are thus not simplicial. Simplicial maps are therefore more restrictive than continuous maps. However, the difference of the two concepts is smaller than one might think at first glance.

Fact 3.13. *Every continuous map* $f : |K_1| \to |K_2|$ *can be approximated arbitrarily closely by simplicial maps on appropriate subdivisions of* K_1 *and* K_2*.*

This shows that we can consider simplicial maps to be the analogue of continuous maps in the world of simplicial complexes. This begs the question whether other definitions from topology, such as homotopies or deformation retracts, have simplicial analogues. As we will see in the next few definitions, they do.

Definition 3.14. *Two simplicial maps* $f_1, f_2 : K_1 \to K_2$ *are* contiguous *if for every simplex* $\sigma \in K_1$ *we have that* $f_1(\sigma) \cup f_2(\sigma)$ *is a simplex in* K_2*.*

This is the simplicial analogue of two continuous maps being homotopic. We can thus show two simplicial complexes to be homotopy equivalent by providing two simplicial maps $f : K_1 \to K_2$ and $g : K_2 \to K_1$ such that $g \circ f$ is contiguous with the identity map on K_1 and $f \circ g$ is contiguous with the identity map on K_2.

Definition 3.15. *A face of a simplicial complex is called* free*, if it is non-maximal (not inclusion-maximal) and contained in a unique maximal face.*

Note that every face that is a superset of a free face is either a maximal face or also free.

Definition 3.16. *A* collapse *is the operation of removing all faces* γ *that are a superset of some fixed free face* τ *(including* τ *itself). A simplicial complex is* collapsible *if there is a sequence of collapses leading to a single point.*

A collapse can be written as a deformation retract. Thus, a simplicial complex that is collapsible is contractible, and we consider collapses to be the simplicial analogue of deformation retracts.

You might wonder whether every contractible simplicial complex is also collapsible. We will see that this not hold: A good counterexample for this is Bing's house with two rooms, see Figure 3.3. In any triangulation of it, there are no free faces: As a 2-dimensional space, there are only vertices, edges and triangles. We only have to check edges, since triangles are maximal, and vertices are part of edges which are never maximal. Every edge is incident to at least two triangles (there are no edges on the "boundary"), and thus they are not free. Since we have no free faces, it is not collapsible.

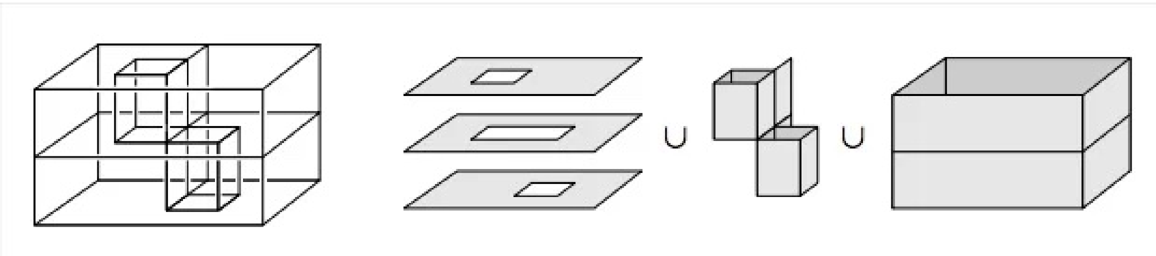


Figure 3.3: *Bing's house with two rooms. Image taken from [2].*

On the other hand, Bing's house is contractible: both Bing's house and a point are deformation retracts of a 3-dimensional ball, and thus by Fact 2.24 they are homotopy equivalent. For a visual sketch of the deformation retract from a 3-dimensional ball to Bing's house, see Figure 3.4.

To summarize, the connection between simplicial complexes and topological spaces is that every simplicial complex defines a topological space, since we can consider a geometric embedding, and the underlying space of the embedding inherits the subspace topology from $\mathbb{R}^d$. On the other hand, some topological spaces (the triangulable ones) can be expressed by simplicial complexes. As for maps, every simplicial map is continuous. On the other hand, continuous maps between simplicial complexes can be approximated by simplicial maps between subdivisions of the simplicial complexes. A similar property holds between homotopic maps and contiguous maps, as well as between deformation retracts and collapses. In general, we can say that the terms in combinatorial topology are special cases of their "continuous" counterparts, and if we consider triangulable spaces, the continuous terms can be approximated in some way by their combinatorial counterparts. The terms can thus be considered to be equivalent.

Table 3.1 summarizes the equivalent words in "continuous topology" on triangulable spaces and in combinatorial topology on simplicial complexes.

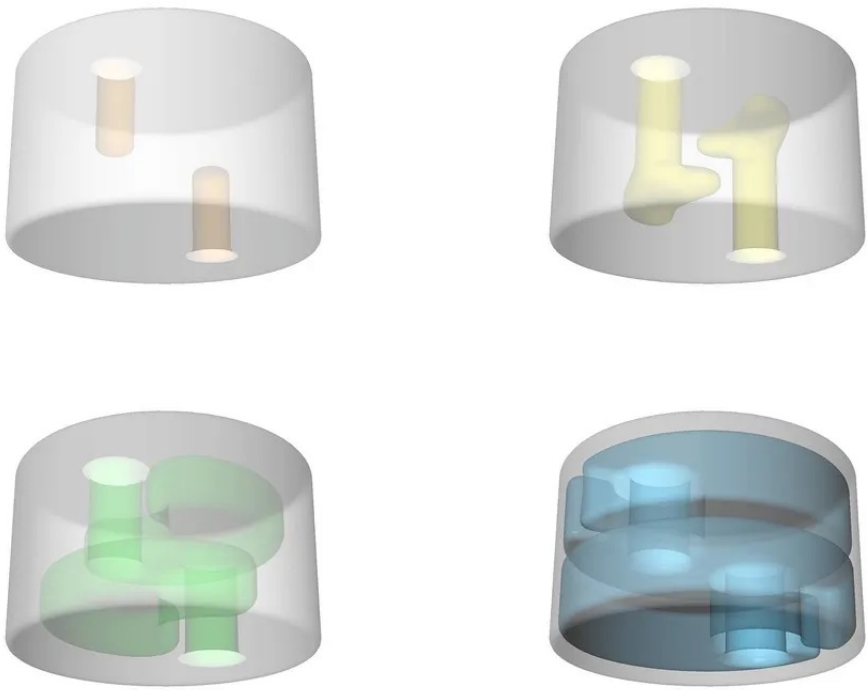


Figure 3.4: *A visual representation of the deformation retract from a 3-dimensional ball to Bing's house. Images taken from the blog Sketches of topology [1].*

"continuous" topology	combinatorial topology
topological spaces	simplicial complexes
continuous maps	simplicial maps
homotopic maps	contiguous maps
deformation retracts	collapses

Table 3.1: *Equivalent notions in "continuous" and combinatorial topology*

3.2 Homology

Recall that homology is intended as a tool to count holes in objects, and recall that this hole count is intended as an invariant of topological spaces under homotopy equivalence. We have introduced simplicial complexes, which allow us to consider concrete combinatorial descriptions instead of abstract topological spaces.

Let us begin with some basic intuition for holes in simplicial complexes, before diving into the more technical definitions. Consider the two simplicial complexes shown in Figure 3.5. How many (and what kind of) holes should these complexes intuitively have?

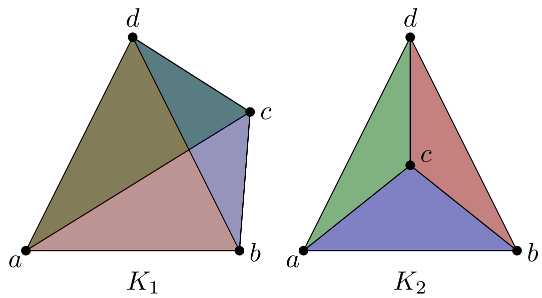


Figure 3.5: *Two simplicial complexes.* K_1 *contains all four triangles* $\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$ *as well as their subsets, while* K_2 *only contains the three triangles* $\{a,b,c\},\{a,c,d\},\{b,c,d\}$ *and their subsets.*

As can be seen, K_1 is the boundary of a tetrahedron. It is a triangulation of the 2-dimensional (hollow) sphere, so we would like to say that it has a hole, or *cavity*. In particular, because this cavity is of the same dimension as the cavity in the 2-dimensional sphere, we want to call this cavity a 2-dimensional hole.

On the other hand, K_2 can be viewed as a triangulation of four points in the plane, where the point a lies inside the convex hull of the other three points. It is homeomorphic to a 2-dimensional disk. Intuitively we would like to say that the complex K_2 does not have any holes.

As a 2-dimensional disk, K_2 has a *boundary*, consisting of the edges $\{a,b\}$, $\{b,d\}$ and $\{a,d\}$. On the other hand, K_1 has no boundary, just as a sphere has no boundary. We will later define a notion of boundary capturing this intuition, at least for *pure* simplicial complexes, that is, simplicial complexes whose maximal faces all have the same dimension. For example, a 1-dimensional pure simplicial complex is just a graph with no isolated vertices. In such a graph, the boundary will contain all the leaves (vertices of degree 1). Some complexes, like K_1, will have an empty boundary, and in analogy to graphs without leaves we call such complexes *cycles*[1]. Under this viewpoint, our d-dimensional holes of a simplicial complex K should be d-dimensional pure subcomplexes that are cycles. On the other hand, clearly not all cycles should be holes, as can be seen with the boundary of K_2. This boundary (the three edges $\{a,b\}$, $\{b,d\}$ and $\{a,d\}$) itself does not have a boundary, and is thus a 1-dimensional cycle. However we do not want to consider this cycle as a 1-dimensional hole of K_2 since it is "filled up", it is the boundary of the three filled in triangles.

Summed up, our intuition is that holes are subcomplexes that have no boundary (cycles) and that are not themselves boundaries of another subcomplex which would be filling in the hole. In the following we will make this intuition precise by defining the types of subcomplexes we consider, the notions of boundaries and cycles, and how to mathematically describe the cycles that are not boundaries.

[1]Note that technically graphs without leaves are not necessarily just cycles, but can also consist of multiple cycles glued together at vertices and edges.

3.2.1 Chains

In the following we let K be a simplicial complex, and we use m_p to denote the number of p-simplices in K. We first want to define p-*chains*, which are simply an algebraic way of formalizing and generalizing subsets of p-simplices.

Definition 3.17. *A* p-chain c (in K) *is a formal sum of* p*-simplices added with some coefficients from some ring* R*. A* p*-chain* c *can thus be written as*

$$c = \sum_{i=1}^{m_p} \alpha_i \sigma_i,$$

where $\alpha_i \in R$ *and* $\sigma_i \in K$ *are* p*-simplices.*

All we are doing in this formal sum is giving a coefficient from R to each p-simplex of K. A formal sum is only a sum in a syntactic sense (i.e., we use the symbols $+$ and $\sum$), but there is no semantic meaning behind this operation; there is no other way to represent a chain other than the sum it is defined by.

Using the addition operation of the ring R however, we can now also add two p-chains $c = \sum \alpha_i \sigma_i$ and $c' = \sum \alpha_i' \sigma_i$ (both in K). Since the chains are both just formal sums, we can simply do this addition "component-wise", using addition in R on the coefficients:

$$c + c' := \sum_{i=1}^{m_p} (\alpha_i + \alpha_i')\sigma_i$$

We therefore have an addition operation on the set $C_p(K)$ of all p-chains in K. We show next that $C_p(K)$ endowed with this operation forms a group, and we call it the p*-th chain group* (of K).

Observation 3.18. $(C_p(K), +)$ *is an abelian group, it is free, and the* p*-simplices form a basis.*

Proof. To show that it is a group we observe that:

1. $C_p(K)$ is closed under addition, since $\forall c_1, c_2 \in C_p(K)$ we have $c_1 + c_2 \in C_p(K)$.

2. The operation $+$ is associative: $\forall c_1, c_2, c_3 \in C_p(K)$, $(c_1 + c_2) + c_3 = \sum(\alpha_i^{(1)} + \alpha_i^{(2)})\sigma_i + \sum \alpha_i^{(3)}\sigma_i = \sum(\alpha_i^{(1)} + \alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i = \sum \alpha_i^{(1)}\sigma_i + \sum(\alpha_i^{(2)} + \alpha_i^{(3)})\sigma_i = c_1 + (c_2 + c_3)$.

3. We have a neutral element $0 = \sum 0\sigma_i \in C_p(K)$.

4. Every element has an inverse: $\forall c \in C_p(K)$ we have $-c = \sum(-\alpha_i\sigma_i) \in C_p(K)$ and $c + (-c) = \sum(\alpha_i - \alpha_i)\sigma_i = 0$.

Commutativity follows from $+$ in R being commutative (recall that for any ring $(R, +, \cdot)$, $(R, +)$ is an abelian group), thus the group is abelian. Finally, the p-simplices clearly form a basis since the set of chains is defined as the set of formal sums of these p-simplices. □

We can further turn $C_p(K)$ into an R-module:

Observation 3.19. *Equipped with the appropriate function* $\cdot : R \times C_p(K) \to C_p(K)$, $C_p(K)$ *is an* R*-module.*

Proof (sketch). We can define $r \cdot c$ by simply using the multiplication $\cdot$ of R component-wise on each coefficient of c, i.e., $r \cdot \sum_{i=1}^{m_p} \alpha_i \sigma_i = \sum_{i=1}^{m_p} (r \cdot \alpha_i)\sigma_i$. We leave the proof of the necessary properties as an exercise. □

From now on we will always work with one of the simplest possible rings, the ring $R = \mathbb{Z}_2$. In particular this allows us to simply view chains as sets of p-simplices, the sum of chains being their symmetric differences, and we get the nice identity $c + c = 0$. With $R = \mathbb{Z}_2$, we will define *homology over* $\mathbb{Z}_2$, often also just called $\mathbb{Z}_2$-homology. Using some slightly more abstract definitions, all of the following can be extended to define homology over any ring R. For more on this, we refer to any textbook on algebraic topology, e.g., the one by Hatcher [2].

3.2.2 Boundary Maps

Now that we can talk algebraically about sets of p-simplices, we can now formalize the notion of the boundary. It should be intuitively clear what the boundary of a single p-simplex should be: just take the $(p-1)$-chain formed by its facets.

More formally, let $\sigma = \{v_0, \dots, v_p\}$ be a p-simplex. Then, $\delta_p(\sigma)$ is defined by

$$\{v_1, \dots, v_p\} + \{v_0, v_2, \dots, v_p\} + \dots + \{v_0, \dots, v_{p-1}\} = \sum_{i=0}^{p} \{v_0, \dots, \hat{v_i}, \dots, v_p\}$$

In the above notation, $\hat{v_i}$ denotes that the element v_i is omitted from the set. Note that $\delta_p(\sigma)$ is indeed a $(p-1)$-chain. For some examples, see Figure 3.6.

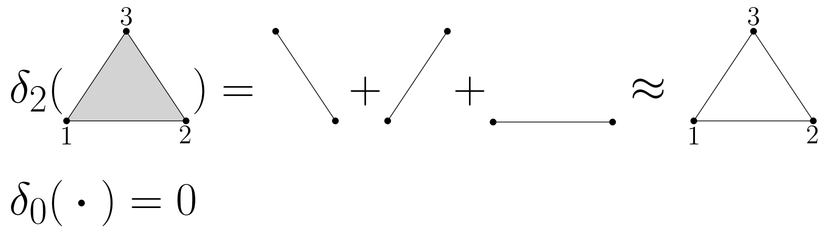
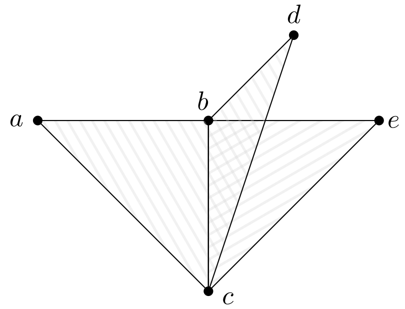


Figure 3.6: *The boundary chains of two different simplices.*

We have seen that δ_p is a map that sends a p-simplex to a $(p-1)$-chain. Thanks to the group structure of the chain group, we can now immediately extend this to any chain. After this extension, δ_p defines a map from $C_p(K)$ to $C_{p-1}(K)$:

$$\begin{aligned} \delta_p : C_p(K) &\to C_{p-1}(K) \\ c = \sum \alpha_i \sigma_i &\mapsto \delta_p(c) = \sum \alpha_i(\delta_p(\sigma_i)) \end{aligned}$$

It is easy to prove that δ is a group homomorphism, and we call it the *boundary operator* homomorphism. Let us apply this definition to the following example. In a slight abuse of notation in favor of legibility, we denote faces $\{a, b, c\}$ by abc.



$$\begin{aligned} \delta_2(abc + bcd) &= \delta_2(abc) + \delta_2(bcd) \\ &= (ab + bc + ac) + (bc + cd + bd) \\ &= ab + ac + cd + bd \end{aligned}$$

$$\begin{aligned} \delta_2(abc + bcd + bce) &= (ab + bc + ac) + (bc + cd + bd) + (bc + ce + be) \\ &= ab + bc + ac + cd + bd + ce + be \end{aligned}$$

We can see that an edge is in the boundary of a chain of triangles exactly if it is contained in an odd number of triangles of the chain, thanks to the fact that we use addition in $\mathbb{Z}_2$.

We have already seen that cycles can be boundaries. On the flipside we have also seen that the boundary of a simplex should have no boundary (i.e., it should be a cycle), where the interior of the simplex fills up the cavity given by its boundary. The following lemma generalizes this to boundaries of any chain: It states that the boundary of any boundary is empty.

Lemma 3.20. *For* $p > 0$, $\delta_{p-1} \circ \delta_p(c) = 0$, *for any* p-*chain* c.

In the example above, $\delta_1(\delta_2(abc + bcd)) = (a + b) + (a + c) + (c + d) + (b + d) = 0$.

Proof. It is enough to show this for simplices, as $\delta_{p-1} \circ \delta_p(c) = \delta_{p-1}(\sum \alpha_i(\delta_p(\sigma_i))) = \sum \alpha_i(\delta_{p-1} \circ \delta_p(\sigma_i))$. For a p-simplex σ, every $(p-2)$-face of σ is contained in exactly 2 $(p-1)$-faces of σ, and does thus not appear in $\delta_{p-1} \circ \delta_p(\sigma)$. □

The notions of homology we will introduce below actually generalize to any sequence of group homomorphisms δ_i that fulfill Lemma 3.20 above. Each such sequence of homomorphisms defines a so-called *chain complex*:

$$0 = C_{k+1}(K) \xrightarrow{\delta_{k+1}} C_k(K) \xrightarrow{\delta_k} C_{k-1}(K) \cdots C_2(K) \xrightarrow{\delta_2} C_1(K) \xrightarrow{\delta_1} C_0(K) \xrightarrow{\delta_0} C_{-1} = 0$$

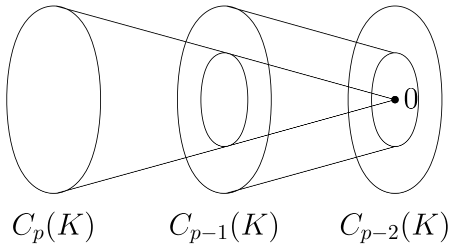


Figure 3.7: *A schematic illustration of a part of a chain complex.*

3.2.3 Cycle and Boundary Groups

As we already established intuitively, chains without boundaries are called cycles. These are the objects potentially giving rise to holes or cavities.

Definition 3.21. *A* p-*chain* c *is a* p-cycle *if* $\delta(c) = 0$. $Z_p(K)$ *is the* p-th cycle group, *consisting of all* p-*cycles of* K.

Lemma 3.22. $Z_p(K)$ *is a group.*

Proof. $Z_p(K) = \ker \delta_p$. (Recall that the kernel of a homomorphism is a subgroup of its domain.) □

So far we have only formally defined a boundary operator, but have not specified which chains we call boundaries. Of course, as already used implicitly before, the boundaries are the chains that are the result of applying the boundary operator.

Definition 3.23. *A* p-*chain* c *is a* p-boundary *if* $\exists c' \in C_{p+1}(K)$ *such that* $\delta(c') = c$. $B_p(K)$ *is the* p-th boundary group, *consisting of all* p-*boundaries of* K.

Lemma 3.24. $B_p(K)$ *is a group.*

Proof. $B_p(K) = \operatorname{im} \delta_{p+1}$. □

In the following, we will often drop the "(K)" of $C_p(K), Z_p(K)$, and $B_p(K)$ when it is clear which simplicial complex we are speaking about.

Fact 3.25. $B_p \subseteq Z_p \subseteq C_p$, *and all of them are abelian and free.*

We will not prove this statement here, but to see that $B_p \subseteq Z_p$, recall that by Lemma 3.20 the boundary of a boundary is empty.

3.2.4 Homology Groups

We are now ready to formalize the notion of holes or cavities. Recall that intuitively, a hole is a cycle that is not a boundary, that is, not filled by something higher-dimensional. Using that all objects defined so far form abelian groups, we can phrase this in algebraic terms using quotient groups.

Definition 3.26. *The* p-th homology group $H_p(K)$ *is the quotient group* $Z_p(K)/B_p(K)$.

Often in the literature we write $H_p(K; R)$ for homology over some ring R. Since we only work with homology over $\mathbb{Z}_2$ in these lecture notes, we just write $H_p(K)$ to mean $H_p(K; \mathbb{Z}_2)$.

Remember that the elements of a quotient group are *cosets*. In essence, each element of the homology group is a coset called a *homology class* which contains cycles that differ only by boundaries. The coset $[c] = c + B_p$ is the homology class of c. We say that c and c' are *homologous*, if $[c] = [c']$, which is equivalent to the statements $c \in c' + B_p$ and $c + c' \in B_p$. See Figure 3.8 for an example of homologous cycles, and Figure 3.9 for an example of the first homology group of a small complex.

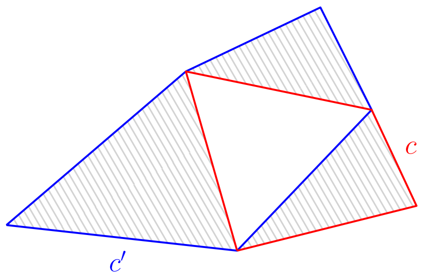


Figure 3.8: c' *and* c *are homologous cycles.*

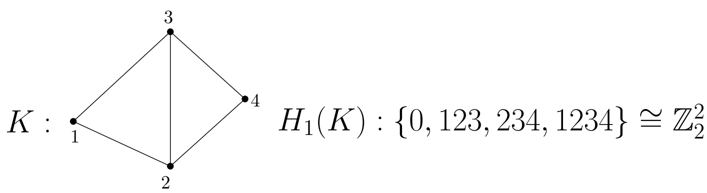


Figure 3.9: *The first homology group of a small complex.*

Exercise 3.27. *Visualize the following simplicial complex* K*: 0-faces* $\{a, b, c, d, e\}$*, 1-faces* $\{ab, ac, ad, bc, bd, cd, ce, de\}$ *and 2-faces* $\{abc, abd, acd, bcd\}$*. For the dimensions 1 & 2, what are the cycle, boundary, and homology groups of* K*? Note: You can express the groups by their generators. You do not need to write out all the elements.*

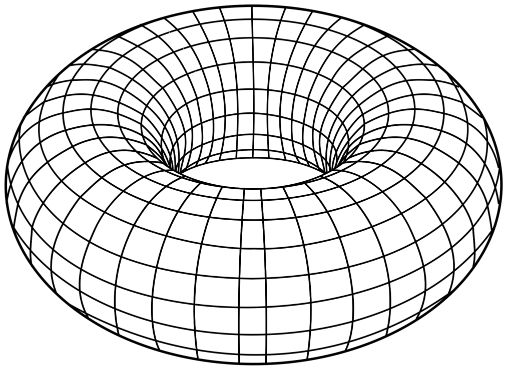


Figure 3.10: *A torus.*

Exercise 3.28. *Give an informal derivation for the homology groups of a torus (see Figure 3.10). Can you find a space with isomorphic homology that is not homeomorphic to the torus?*

Exercise 3.29. *For a simplicial complex* K*, its cone* CK *is the complex with the same set of vertices plus one additional vertex* z*, and such that for all simplices in* K *we have*

$$\{a, b, c, \ldots\} \in \mathsf{K} \implies \{a, b, c, \ldots, z\} \in \mathsf{CK}$$

(a) Visualize a cone operation. What does it intuitively do to a complex?

(b) Show that the homology of the cone CK *is* 0 *in all dimensions* $d > 0$*, for any* K*.*

(c) Bonus: What would happen (intuitively and to the homology) if we extended K *in the same way as before, but with two points? (this is called the* suspension *of* K*)*

Here are some nice properties of homology groups, that will be beneficial for us, but that we will not prove here.

Fact 3.30.

- H_p *is abelian and free.*
- H_p *is a* $\mathbb{Z}_2$*-vector space.*

Remark 3.31. *If we consider homology defined over other rings, e.g. over* $\mathbb{Z}$ *instead of* $\mathbb{Z}_2$*, the homology groups might not be free.*

Recall that our original motivation for introducing homology was to count the number of holes. With homology as we defined it, we have the algebraic structure of a vector space where we can add holes together. The number of distinct holes is now just the dimension of this vector space.

Definition 3.32. $\beta_p := \dim H_p = \dim Z_p - \dim B_p$ *is the* p-th Betti number.

In the definition above, dim denotes the dimension of a vector space as you know it from linear algebra, i.e., $\dim H_p$ is the number of elements in a basis of H_p.

Exercise 3.33. *The* **Euler characteristic** *of a simplicial complex* K *is defined as*

$$\chi = k_0 - k_1 + k_2 - \dots$$

with k_i *denoting the number of i-dimensional simplices in* K*. Convince yourself that this is an invariant property for all triangulations of the same topological space* X*.*

Hint: *Show instead that* $\chi = \beta_0(K) - \beta_1(K) + \dots$*. The statement then follows by the fact that homeomorphic spaces have the same homology.*

Exercise 3.34. *Take any vector* $v = (a_0, \dots, a_d) \in \mathbb{N}^{d+1}$ *with* $a_0 > 0$*. Show that there exists a simplicial complex* K_v *with that vector as its Betti numbers.*

3.2.5 Singular Homology

With our definition of homology for simplicial complexes, we get for free a notion of homology for many topological spaces, namely the triangulable ones: we can simply triangulate them and take the homology of the triangulation. But, a topological space may have many triangulations, and it seems like the structure of the homology groups might differ depending on the choice of triangulation. The aim of this section is to sketch the tools that show that the homology of a triangulable space is independent of the chosen triangulation. The idea of singular homology is to remove the need for a fixed triangulation by looking at *all possible simplices* at once.

Let X be a topological space, and let Δ^p be the standard p-simplex in $\mathbb{R}^{p+1}$. We want to consider all possible occurrences of this simplex in X.

Definition 3.35. *A* singular p-simplex *is a map* $\sigma : \Delta^p \to X$.

Note that in this definition we do not require σ to be injective, thus it would even be possible to map the simplex to a single point.

We now define $C_p(X)$ the same way as before, but now on the family of all singular p-simplices, which in general makes the group uncountably infinite. We also define δ_p as before, defining $Z_p(X)$ and $B_p(X)$, which are now also uncountably infinite. Finally, we again define $H_p(X) = Z_p(X)/B_p(X)$. Surprisingly, this definition agrees with the simplicial definition of homology on any triangulation of X.

Theorem 3.36. *Let* X *be a topological space,* K *a triangulation of* X*. Then we have* $H_p(X) \cong H_p(K)$ *for all* $p \geqslant 0$*.*

As isomorphisms for vector spaces are an equivalence relation, we also get the desired independence of the triangulation.

Corollary 3.37. *Let* K_1, K_2 *be two distinct triangulations of* X*. Then,* $H_p(K_1) \cong H_p(K_2)$ *for all* $p \geqslant 0$*, that is, homology is independent of the chosen triangulation.*

For the remainder of these lecture notes we will only work with simplicial homology, but we often talk about the homology of a triangulable space without specifying a triangulation. The above corollary gives us the right to do this.

3.2.6 The 0-th homology group

The homology group that is easiest to understand is the 0-th one. Recall that the 0-simplices of a simplicial complex K are simply its vertices. Since vertices do not have any boundaries, every vertex is a 0-cycle. The boundary of a 1-simplex simply consists of the two vertices which are connected by the edge. We can thus see that two vertices v_1 and v_2 are homologous if there is a path from v_1 to v_2, and the homology class $[v_1]$ is simply the connected component containing v_1.

Observation 3.38. $\beta_0(K)$ *is the number of connected components of* K*.*

As a consequence, the 0-homology classes are all the formal sums of connected components.

3.2.7 Homology of Spheres

One of the main intuitions for us when we introduced homology was that a d-sphere should have a single d-hole and no other holes. We will now check whether our definition captured this intuition correctly. Since we have seen in Section 3.2.5 that homology is independent from the chosen triangulation, let us fix some triangulation of the sphere S^d. A good candidate (due to its simplicity) is the boundary of a simplex, that is, $S^d \simeq \delta(\Delta^{d+1})$, with the vertex set $V = \{v_0, \dots, v_{d+1}\}$.

$H_0(S^d)$: Let us first investigate $H_0(S^d)$. Since all vertices are connected, all vertices are homologous, and $H_0(S^d) = \langle [v] \rangle \cong \mathbb{Z}_2$.

$H_d(S^d)$: Next, let us check $H_d(S^d)$. We first compute Z_d: The d-simplices are exactly the sets $\sigma_i = \{v_0, \dots, \hat{v_i}, \dots, v_{d+1}\}$. Note that every $(d-1)$-simplex occurs as the boundary of exactly two such d-simplices. Thus, both the zero element (empty chain) as well as the chain c consisting of all d-simplices are part of Z_d. On the other hand, no chain $c' \notin \{0, c\}$ can be a cycle, since for such a chain there must be some d-simplex $\sigma \in c'$

neighboring some d-simplex $\sigma' \notin c'$. The $(d-1)$-simplex that is a boundary of both σ and σ' would then be part of $\delta_p(c')$. We conclude that $Z_d(S^d) = \langle c \rangle$.

Since $\delta(\Delta^{d+1})$ is a d-dimensional simplicial complex, and thus does not contain any $(d+1)$-simplices, no non-empty d-chain can be a boundary. We thus get that $B_d(S^d)$ is the group containing only 0.

We finally get $H_d(S^d) = Z_d/B_d = Z_d \cong \mathbb{Z}_2$.

$H_p(S^d)$: Finally, let us go to $H_p(S^d)$, for $0 < p < d$: Let $c = \sum \alpha_i \sigma_i$ be a p-cycle. We aim to show that c is homologous to the 0-chain, i.e., that $[c] = [0]$. Equivalently, we show that c must be a boundary.

Let $\sigma = (v_{m_0}, \ldots, v_{m_p})$ be any p-simplex in c which does not include v_0. We will keep replacing such simplices by simplices which do contain v_0, until we have no more simplices not containing v_0.

Let b be the $(p+1)$-simplex $(v_0, v_{m_0}, \ldots, v_{m_p})$. Note that $b \in \delta(\Delta^{d+1})$ and thus $\delta(b)$ is a p-boundary. Also note that σ is in $\delta(b)$. Furthermore, σ is the only p-simplex in $\delta(b)$ which does not contain v_0. We now add $\delta(b)$ to c, to get $c' := c + \delta(b)$. Since we added a boundary, $[c] = [c']$ (i.e., c and c' are homologous). Furthermore, c' contains one fewer p-simplex not containing v_0, when compared to c.

We repeat this process until we reach a cycle c^* in which every p-simplex contains v_0. We now claim that c^* must be the trivial cycle: Assume c^* contains some p-simplex $a = (v_0, v_{a_1}, \ldots, v_{a_p})$. Then, the $(p-1)$-simplex $a' = (v_{a_1}, \ldots, v_{a_p})$ is part of $\delta(a)$. But, a' cannot be part of the boundary of any other p-simplex in c^*, since the only p-simplex containing a' as a face while also containing v_0 is a. Thus, to have an empty boundary, we have $c^* = 0$. By construction, $[c] = [c^*]$, therefore $[c] = 0$ as we aimed to prove.

We have proven that every cycle is homologous to 0, and we can conclude that for all $0 < p < d$, $H_p(S^d) = 0$.

Since S^d is d-dimensional, we do not have any simplices of dimensions $p > d$, and thus $H_p(S^d) = 0$ for $p > d$. Combining all these arguments we conclude the following theorem:

Theorem 3.39. *For any $d > 0$, we have*

$$H_p(S^d) = \begin{cases} \mathbb{Z}_2 & p \in \{0, d\} \\ 0 & \textit{else}. \end{cases}$$

$$\beta_p(S^d) = \begin{cases} 1 & p \in \{0, d\} \\ 0 & \textit{else}. \end{cases}$$

3.2.8 Induced Homology

As usual, now that we have defined some mathematical objects (homology groups) we are also interested in the maps between them. For simplicial complexes we have defined simplicial maps, and we now want to study the effect that simplicial maps have on the homology of a space.

We first extend simplicial maps to the chain groups.

Definition 3.40. *Let* $f: K_1 \to K_2$ *be a simplicial map. This induces a* chain map

$$f_\# : C_p(K_1) \to C_p(K_2)$$
$$c = \sum \alpha_i \sigma_i \mapsto f_\#(c) := \sum \alpha_i \tau_i, \text{ where } \tau_i = \begin{cases} f(\sigma_i) & \text{if } f(\sigma_i) \text{ is } p\text{-simplex in } K_2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(\sigma_i)$ is always a simplex in K_2 since f is a simplicial map, but it could be a simplex of smaller dimension. This is why we need the condition in the above definition of τ_i.

The following can be shown with a bit of work:

- $f_\# \circ \delta = \delta \circ f_\#$
- $f_\#(B_p(K_1)) \subseteq f_\#(Z_p(K_1))$
- $f_\#(Z_p(K_1)) \subseteq Z_p(K_2)$, $f_\#(B_p(K_1)) \subseteq B_p(K_2)$

From this chain map $f_\#$, we now get a well-defined *induced homomorphism* between the homology groups of K_1 and K_2.

Definition 3.41. *Let* f *be a simplicial map and* $f_\#$ *its induced chain map. This induces a homomorphism*

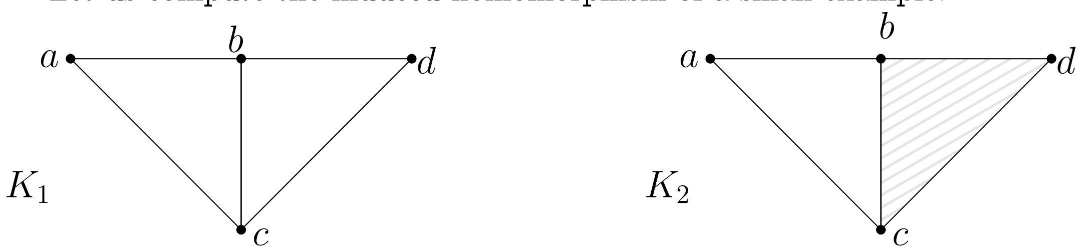
$$f_* : H_p(K_1) \to H_p(K_2)$$
$$[c] = c + B_p \mapsto f_\#(c) + B_p(K_2) = [f_\#(c)].$$

Fact 3.42. *If* $H_p(K_1)$ *and* $H_p(K_2)$ *are vector spaces (as they are in e.g.* $\mathbb{Z}_2$*-homology, which is what we are using), then* f_* *is a linear map.*

We also get the following *functorial property*, which we will not prove.

Fact 3.43. *For two simplicial maps* $f: X \to Y$, $g: Y \to Z$, *we have* $(g \circ f)_* = g_* \circ f_*$.

Let us compute the induced homomorphism of a small example:



We consider the inclusion map $f: K_1 \hookrightarrow K_2$.

$$H_1(K_1) = \{0, [abc], [bcd], [abdc]\} \cong \mathbb{Z}_2^2$$
$$f_*(0) = 0,\ f_*([abc]) = [abc]$$
$$f_*([bcd]) = 0,\ f_*([abdc]) = [abc]$$

Exercise 3.44. *Let*

$$K_1 = \{\emptyset, a, b, c, d, e, ab, ac, bc, bd, cd, ce, de, abc\}$$

and

$$K_2 = \{\emptyset, w, x, y, z, wx, wy, xy, xz, yz\}.$$

Consider the map $f : K_1 \to K_2$ *induced by the vertex map*

$$a \mapsto y, b \mapsto x, c \mapsto y, d \mapsto z, e \mapsto z.$$

You can verify easily that f *is simplicial. Compute* $f_* : H_p(K_1) \to H_p(K_2)$ *for* $0 \leqslant p \leqslant 2$.

Exercise 3.45. *Which of the following four statements is true for every simplicial map* f*?*
"If f *is {injective, surjective}, then* f_* *is {injective, surjective}."*

The following fact has some very powerful consequences, as we will see.

Fact 3.46. *If* $f, g : K_1 \to K_2$ *are contiguous,* $f_* = g_*$.

Note that the definition of induced homology extends from simplicial maps to maps between any topological spaces. We will not state the exact definitions, but the following fact is the continuous analogue (remember that two simplicial maps being contiguous is analogous to two maps being homotopic) of Fact 3.46.

Fact 3.47. *If* $f, g : X \to Y$ *are homotopic,* $f_* = g_*$.

Thanks to this fact we get the following corollary, which shows that homology is indeed an invariant under homeomorphisms, and even under homotopy equivalence. This also gives us the option to compute the homology of a space by computing the homology of a potentially simpler homotopy equivalent space.

Corollary 3.48. *If* $f : X \to Y$ *is a homotopy equivalence (i.e., there exists* $g : Y \to X$ *such that* $f \circ g$ *is homotopic to* id_Y *and* $g \circ f$ *is homotopic to* id_X*), then* f_* *is an isomorphism.*

Proof. Thanks to Fact 3.43 we have $(g \circ f)_* = g_* \circ f_*$. By Fact 3.47, $(g \circ f)_* = (\mathrm{id}_X)_*$, which is an isomorphism. Since we thus know that $g_* \circ f_*$ is an isomorphism, we know that f_* must be injective and g_* must be surjective. By a symmetric argument considering $f \circ g$ we also get that f_* is surjective and g_* is injective, and thus both f_* and g_* are isomorphisms. □

Exercise 3.49.

Consider the space you get when you glue together two points of a torus. What is the homology of this space?

Consider the space you get when you simultaneously pierce a balloon at n distinct locations. What is the homology of this space?

Exercise 3.50. *Let $f, g : S^1 \to S^1$ be continuous maps such that $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all $x \in S^1$.*

a) Convince yourself that $f_* : H_1(S^1) \to H_1(S^1)$ is trivial (maps everything to 0) and that g_* is an isomorphism.

b) Show that f and g are not homotopic.

c) Show that there is no map $h : S^2 \to S^1$ such that $h(-x) = -h(x)$.

d) Conclude that every map $\phi : S^2 \to \mathbb{R}^2$ with $\phi(-x) = -\phi(x)$ has a zero.

The statement you have proven in d) is equivalent to the 2-dimensional case of the famous Borsuk-Ulam theorem, which implies statements such as "at any time, there are two antipodal points on the earth with both the same temperature and atmospheric pressure".

3.2.9 Application: Brouwer Fixed Point Theorem

In this section we finally collect the fruits of our hard work by using homology to give a relatively short proof of the famous fixed point theorem by Brouwer. Here, $\mathbb{B}^d$ denotes the unit ball of dimension d.

Theorem 3.51 (Brouwer fixed point theorem). *Let $f : \mathbb{B}^d \to \mathbb{B}^d$ be continuous. Then, f has a fixed point, that is, $\exists x \in \mathbb{B}^d$ such that $f(x) = x$.*

This theorem has many fascinating implications:

- Take two sheets of paper lying on top of each other. Crumple the top sheet and set it back onto the other sheet. No matter how you crumpled the sheet, at least one point of the crumpled sheet lies exactly above its corresponding point in the bottom sheet.
- If you open a map of Switzerland in Switzerland, there is at least one point on the map which is at its exact position.[2]
- If you take a cup of liquid and stir or slosh it, at least one atom ends up at its original position (but if you shake you might break continuity).
- The theorem also has many applications in mathematics and computer science, such as in fair divisions or for proving existence of Nash equilibria.

[2] The theorem only applies when ignoring the Italian and German exclaves of Campione d'Italia and Büsingen am Hochrhein.

To prove Theorem 3.51, we first introduce the following definition and a helper lemma, which we only prove after proving Theorem 3.51 itself.

Definition 3.52. *A map* $r : X \to A \subseteq X$ *is a* retraction *if* $r(a) = a$, $\forall a \in A$.

Lemma 3.53. *There is no retraction* $r : \mathbb{B}^d \to S^{d-1}$.

Proof of Theorem 3.51. We prove the theorem by contradiction. For an illustration of the argument see Figure 3.11. Assume $f : \mathbb{B}^d \to \mathbb{B}^d$ has no fixed point. For each x, consider the ray $\overrightarrow{f(x)x}$ and let $r(x)$ be the intersection of this ray with S^{d-1}. Then, $r : \mathbb{B}^d \to S^{d-1}$ is continuous (which we do not prove here) and $r(s) = s\ \forall s \in S^{d-1}$, since no matter where $f(s)$ lies, $\overrightarrow{f(s)s}$ first intersects S^{d-1} in s. Thus, r is a retraction, which does not exist by Lemma 3.53. □

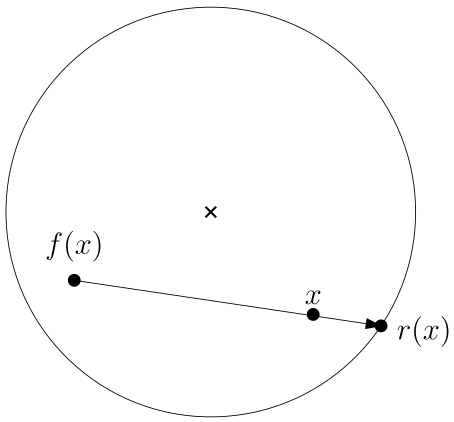


Figure 3.11: *If* f *has no fixed point, we get a retraction to the boundary.*

It remains to prove the helper lemma.

Proof of Lemma 3.53. Consider the inclusion map $i : S^{d-1} \hookrightarrow \mathbb{B}^d$, and a retraction $r : \mathbb{B}^d \to S^{d-1}$. By definition, we have $r \circ i = \text{id}$. Let us look at the induced maps of r and i in the $(d-1)$-th homology of S^{d-1} and $\mathbb{B}^d$. Recall that $H_{d-1}(S^{d-1}) \cong \mathbb{Z}_2$ and $H_{d-1}(\mathbb{B}^d) \cong 0$. We thus view i_* as a homomorphism from $\mathbb{Z}_2$ to 0, and r_* as a homomorphism from 0 to $\mathbb{Z}_2$. But since $r \circ i = \text{id}$, we also have $r_* \circ i_* = \text{id}$. We can combine this to reach a contradiction:

$$1 = \text{id}(1) = (r_* \circ i_*)(1) = r_*(i_*(1)) = r_*(0) = 0$$

Thus, either i or r cannot exist, but since i exists, r cannot. □

Questions

5. *What is a simplicial complex?* Define geometric and abstract simplicial complexes and state and prove the realization theorem (Theorem 3.5).
6. *What are simplicial and contiguous maps?* State the definitions and discuss the connection to their counterparts in continuous topology.
7. *Is every contractible simplicial complex collapsible?* Define the notion of collapsibility and describe Bing's house with two rooms.
8. *What is simplicial homology?* Explain the intuition and give the formal definitions of chains, boundaries and cycles.
9. *Why is the homology of a triangulable space independent of the chosen triangulation?* Explain the idea of singular homology.
10. *What are the homology groups of a sphere?* State and prove the corresponding theorem (Theorem 3.39).
11. *How does a simplicial map between two simplicial complexes induce maps between their homology groups?* Define induced homomorphisms.
12. *What is the Brouwer fixed point theorem?* State, illustrate and prove the Brouwer fixed point theorem (Theorem 3.51).

References

[1] Kenneth Baker, Sketches of topology - Bing's house. https://sketchesoftopology.wordpress.com/2010/03/25/bings-house/, accessed: 2023-04-27.

[2] Allen Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2000.