# Chapter 9

# Multiparameter persistence

*This chapter closely follows the introductory paper [2] on multiparameter persistence. The interested reader is referred there for a (much) more comprehensive overview of the topic.*

As we have seen, the persistence modules arising from the Čech or Vietoris-Rips complexes are *stable* under small perturbations of the underlying data, but not *robust*. That is, even a small number of outliers can drastically change the persistence diagram or barcode. We could try to remedy this problem by making these complexes *density-aware* in some way. For the Vietoris-Rips complex, a typical way to do this is as follows. For a vertex $v$ in a simplicial complex, its degree $\deg(v)$ is the number of edges (1-simplices) in the complex that contain $v$. Then. for $d \in \mathbb{N}$, the degree-$d$ Vietoris-Rips complex is

$$\mathbb{VR}^r_d(X) := \{\sigma \in \mathbb{VR}^r(X) : \text{ each vertex of } \sigma \text{ has degree at least } d\}.$$

Note that vertices corresponding to data points in high-density areas of $X$ will have relatively higher degree, whereas outliers will have relatively lower degree. Thus, for $d$ large enough, we should expect this modification to reduce the impact of outliers. On the other hand, if $d$ is too large, we are 'throwing away the baby with the bathwater'. So, the question is: how to choose $d$? Here, we run into the same issue that originally motivated persistence: there might not be one choice of $d$ that accurately reflects the entire data set. Even if this choice would exist, it might be hard to determine. Instead, we would like to consider all choices of $d$ simultaneously. The solution is to look at persistence w.r.t. both the *scale parameter* $r$ and the *density parameter* $d$ at the same time. In this chapter, we formalize such *multiparameter* persistence, and take a look at the representability and robustness of the resulting *multiparameter persistence modules*.

## 9.1 Persistence modules indexed by a poset

A persistence module (indexed by $\mathbb{R}$) consists of a family of vectors spaces $U_a$, $a \in \mathbb{R}$, together with commuting maps $U_a \to U_b$ for $a \leqslant b$. If we want to define persistence modules indexed by more exotic sets, we need to first extend the meaning of '$\leqslant$'.

**Definition 9.1.** *A* poset *(partially ordered set) is a tuple* $(P, \preceq)$ *consisting of a set* $P$ *and a binary relation* $\preceq$ *on* $P$ *satisfying*

- $a \preceq a$ *for all* $a \in P$*;*
- $a \preceq b$ *and* $b \preceq a$ *implies* $a = b$ *for all* $a, b \in P$*;*
- $a \preceq b$ *and* $b \preceq c$ *implies* $a \preceq c$ *for all* $a, b, c \in P$*;*

*We will sometimes simply write* $P$ *if the (partial) ordering* $\preceq$ *is clear from context.*

Note that partial orderings allow for elements to be *incomparable*, i.e., it can happen for $a, b \in P$ that neither $a \preceq b$ nor $b \preceq a$ holds. In contrast, under a *total ordering* all elements in $P$ must be comparable. All the posets we will see in this chapter are constructed from the following examples:

- The reals $\mathbb{R} = (\mathbb{R}, \leqslant)$ and natural numbers $\mathbb{N} = (\mathbb{N}, \leqslant)$ with their usual ordering are posets.
- If $(P, \preceq)$ is a poset, and $Q \subseteq P$, then $(Q, \preceq)$ is a poset;
- For any poset $(P, \preceq)$, we have the *opposite poset* $P^{op} = (P, \succeq)$, where

  $$a \succeq b \iff b \preceq a \quad \text{for all } a, b \in P.$$

- For two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ we have the *product poset* $(P \times Q, \preceq)$, where

  $$(p, q) \preceq (p', q') \iff p \preceq_P p' \text{ and } q \preceq_Q q' \quad \text{for all } p, p' \in P \text{ and } q, q' \in Q.$$

We can now define filtrations and persistence modules indexed by an arbitrary poset, generalizing our definitions from earlier chapters.

**Definition 9.2.** *Let* $(P, \preceq)$ *be a poset. A family* $(X_p)_{p \in P}$ *of topological spaces (or simplicial complexes) is called a* $P$*-filtration if* $X_p \subseteq X_{p'}$ *for all* $p, p' \in P$ *with* $p \preceq p'$*.*

**Definition 9.3.** *Let* $(P, \preceq)$ *be a poset, and let* $\mathbb{F}$ *be a field. A* $P$*-persistence module (over* $\mathbb{F}$*) consists of*

- *An* $\mathbb{F}$*-vector space* $U_p$ *for each* $p \in P$*;*
- *A linear map* $u_{p,p'} : U_p \to U_{p'}$ *for each* $p, p' \in P$ *with* $p \preceq p'$*, satisfying* $u_{p_2,p_3} \circ u_{p_1,p_2} = u_{p_1,p_3}$ *for all* $p_1, p_2, p_3 \in P$ *with* $p_1 \preceq p_2 \preceq p_3$*.*

**Observation 9.4.** *If* $(X_p)_{p \in P}$ *is a* $P$*-filtration for some poset* $P$*, then taking* $k$*-dimensional homology gives us a* $P$*-persistence module* $H_k(X_p)$ *(where, as before, the maps* $u_{p,p'} : H_k(X_p) \to H_k(X_{p'})$ *are induced by the inclusions* $X_p \hookrightarrow X_{p'}$*).*

We can now give two examples of filtrations indexed by $P = \mathbb{R} \times \mathbb{N}^{op}$, each of which induces a P-persistence module via the observation above. The first one formalizes the discussion at the beginning of the chapter. The second one applies a similar idea to the Čech complex.

**Example 9.5.** *Let* $X$ *be a finite metric space. Then the family* $(\mathbb{VR}_d^r(X))_{r\in\mathbb{R},d\in\mathbb{N}}$ *defined above is a filtration indexed by* $\mathbb{R} \times \mathbb{N}^{op}$*. It is called the* degree-Rips bifiltration*.*

**Example 9.6.** *Let* $X \subseteq \mathbb{R}^n$ *be a finite point cloud. For* $r \in \mathbb{R}$ *and* $d \in \mathbb{N}$*, define*

$$\mathcal{MC}_d^r = \{x \in \mathbb{R}^n : |B(x,r) \cap X| \geqslant d\} \subseteq \mathbb{R}^n.$$

*Note that, for* $d = 1$*, this is just the union-of-balls used to define the Čech complex. The family* $(\mathcal{MC}_d^r(X))_{r\in\mathbb{R},d\in\mathbb{N}}$ *is a filtration (of topological spaces) indexed by* $\mathbb{R}\times\mathbb{N}^{op}$*. It is called the* multicover bifiltration*.*

Both of the filtrations above can also be thought of as indexed over $\mathbb{R}^2$ (after a reparameterization). In general, we call an $\mathbb{R}^n$-filtration (resp. $\mathbb{R}^n$-persistence module) an $n$-parameter filtration (resp. $n$-parameter persistence module). For $n \geqslant 2$, we also say 'multiparameter'. For $n = 2$ the terms bifiltration and bipersistence module are common.

## 9.2 Representing persistence modules indexed by posets

### 9.2.1 Barcodes?

Recall that a p.f.d. persistence module $\mathbb{U}$ indexed by $\mathbb{R}$ can be uniquely represented by a barcode. In Chapter 6, we saw that this follows from the fact that any such $\mathbb{U}$ can be decomposed into interval modules in a unique way:

$$\mathbb{U} \cong \bigoplus_{i\in I} \mathbb{I}_{\langle a,b\rangle}.$$

Each interval in this decomposition corresponds to a bar in the barcode of $\mathbb{U}$. We could hope a similar statement holds for modules indexed by any poset P. The following definition and theorem should give us some hope:

**Definition 9.7.** *A persistence module* $\mathbb{U}$ *is called* indecomposable *if* $\mathbb{U} \cong \mathbb{U}_1 \oplus \mathbb{U}_2$ *implies that* $\mathbb{U}_1 = 0$ *or* $\mathbb{U}_2 = 0$*.*

**Exercise 9.8.** *Show that interval modules (indexed by* $\mathbb{R}$*) are indecomposable.*

**Theorem 9.9.** *Let* $\mathbb{U}$ *be a p.f.d. persistence module (indexed by a poset* $P$*). Then, there is a unique decomposition:*

$$\mathbb{U} \cong \bigoplus_{i\in I} \mathbb{U}_i,$$

*where each persistence module* $\mathbb{U}_i$ *is indecomposable.*

Understanding P-persistence modules thus boils down to understanding indecomposables (indexed by P). For $P = \mathbb{R}$, it turns out that indecomposables are precisely interval modules, leading to Theorem 6.36. However, for general P, the situation is much more complicated. Without going into details: It is not possible to parameterize indecomposables in terms of some nice family of subsets of P. The decomposition of Theorem 9.9 therefore does not lead to a workable notion of barcode in general. In fact, it is not possible to define any reasonable notion of a barcode for P-persistence modules, in the following sense.

**Definition 9.10.** *Let $(P, \preceq)$ be a poset and let $\mathbb{U}$ be a $P$-persistence module. We say a multiset $\mathcal{B}$ of subsets of $P$ is a* reasonable barcode *for $\mathbb{U}$ if*

$$\mathrm{rank}(u_{p,p'}) = |\{B \in \mathcal{B} : p, p' \in B\}| \quad (\forall p \preceq p').$$

*That is, the rank of the map $u_{p,p'} : U_p \to U_{p'}$ can be computed by counting the number of 'bars' that contain both $p$ and $p'$.*

**Exercise 9.11.** *Show that the usual barcode for a p.f.d. persistence module indexed by $\mathbb{R}$ is reasonable in the above sense.*

**Exercise 9.12.** *For $p \in P$, show that* $\dim U_p$ *is greater than or equal to the number of bars that contain $p$ in a reasonable barcode for $\mathbb{U}$.*

**Example 9.13.** *Let $P = \{0,1,2\} \times \{0,1,2\}$ and consider the following persistence module indexed by $P$:*

$$\mathbb{U} = \begin{array}{ccccc} \mathbb{F} & \xrightarrow{\mathrm{id}} & \mathbb{F} & \longrightarrow & 0 \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow \\ \mathbb{F} & \xrightarrow{f} & \mathbb{F}^2 & \xrightarrow{h} & \mathbb{F} \\ \uparrow & & {\scriptstyle j}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow \\ 0 & \longrightarrow & \mathbb{F} & \xrightarrow{\mathrm{id}} & \mathbb{F} \end{array}, \quad \text{where} \quad \begin{array}{l} f : a \mapsto (a, 0) \\ g : (a,b) \mapsto a \\ h : (a,b) \mapsto a + b \\ j : a \mapsto (0, a) \end{array}$$

*We claim $\mathbb{U}$ cannot have a reasonable barcode. To see this, suppose $\mathcal{B}$ is a reasonable barcode for $\mathbb{U}$. Note that*

$$\mathrm{rank}(h \circ f) = \mathrm{rank}(g \circ f) = \mathrm{rank}(h \circ j) = 1.$$

*By the reasonability assumption, there thus must be subsets $I, J, K \in \mathcal{B}$ with*

$$(0,1), (2,1) \in I, \quad (0,1), (1,2) \in J, \quad (1,0), (2,1) \in K.$$

*Since* $\dim U_{0,1} = \dim U_{2,1} = 1$, *we know by Exercise 9.12 that $(0,1)$ and $(2,1)$ occur in at most one element of $\mathcal{B}$. But that means that $I = J$, and $I = K$, and so in fact $I = J = K \supseteq \{(0,1), (2,1), (1,2)\}$. Thus, using reasonability again, we find that*

$$\mathrm{rank}(g \circ j) \geqslant 1,$$

*contradicting the fact that $g \circ j = 0$.*

For completeness, we note that it is possible to have reasonable barcodes for multiparameter persistence modules if we allow 'bars' to occur with *negative* multiplicity. This leads to so-called *signed barcodes*. Furthermore, there are several special classes of multiparameter persistence modules which do admit an interval decomposition (note that we did not formally define what an interval is in the poset-setting!). This is true in particular for so-called *interlevel persistence*. We will not discuss this further here.

### 9.2.2 Other representations and visualizations

Even though we cannot hope for a barcode-like representation of multiparameter persistence modules, there are still several useful (partial) representations available. We mention just three of them here:

- The *Hilbert-function* of a p.f.d. persistence module $\mathbb{U}$ indexed by $P$ sends $p \in P$ to $\dim U_p \in \mathbb{N}$. It can be thought of as a 'homological heatmap'.

- The *rank-invariant* of $\mathbb{U}$ sends a pair $(p, p')$, $p \preceq p'$ to the rank $\operatorname{rank}(u_{p,p'})$ of the map between $U_p$ and $U'_p$. Note that any two modules with $\mathbb{U} \cong \mathbb{V}$ have the same rank invariant. For modules indexed by $\mathbb{R}$, this implication also holds in the other direction. However, for general $P$, there exist nonisomorphic modules with the same rank-invariant;

- Let $L : t \mapsto ta + b$ be an affine line in $\mathbb{R}^2$ with non-negative slope. Then the restriction $\mathbb{U}_L$ of a persistence module $\mathbb{U}$ indexed by $\mathbb{R}^2$ to the line $L$ can be viewed as a persistence module indexed by $\mathbb{R}$ (with its usual, total ordering). The *Fibered Barcode* of $\mathbb{U}$ sends each such line $L$ to the barcode $\mathcal{B}(\mathbb{U}_L)$. This idea can be extended to lines in $\mathbb{R}^n$;

**Exercise 9.14.** *Show that the rank-invariant of an $\mathbb{R}^2$-persistence module can be recovered from its fibered barcode, and vice versa.*

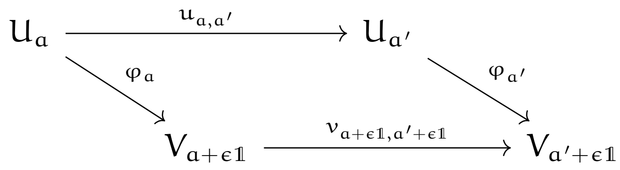
The RIVET software package (`rivet.readthedocs.io`) allows you to compute and visualize each of the invariants above for (among others) the degree-Rips bifiltration.

## 9.3 Distances and robustness for $P$-persistence modules

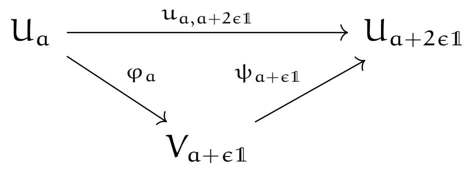
As we generally do not have barcodes for $P$-persistence modules, we cannot rely on the Bottleneck distance to compare them. Instead, we work directly with interleaving distance, which is still well-defined. For simplicity, we only consider $P = \mathbb{R}^n$.

**Definition 9.15.** *Write $\mathbb{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$. Two $\mathbb{R}^n$-persistence modules $\mathbb{U}, \mathbb{V}$ are $\epsilon$-interleaved if there exist two families of linear maps, $\varphi_a : U_a \to V_{a+\epsilon\mathbb{1}}$ and*

$\psi_a : V_a \to U_{a+\epsilon\mathbb{1}}$, $a \in \mathbb{R}^n$, *such that the following diagrams (and their symmetric counterparts obtained by exchanging the role of* $\mathbb{U}$ *and* $\mathbb{V}$*) are commutative:*




*and*




*The interleaving distance between* $\mathbb{U}, \mathbb{V}$ *is* $d_I(\mathbb{U}, \mathbb{V}) := \inf_{\epsilon \geqslant 0}\{\mathbb{U}, \mathbb{V}$ *are* $\epsilon$*-interleaved*$\}$.

The definition above agrees with our earlier definition of interleaving when $n = 1$. Contrary to the 1-dimensional case, computing the interleaving distance for $n \geqslant 2$ is an NP-hard problem (even if the persistence modules $\mathbb{U}, \mathbb{V}$ are given to us in a 'nice' form). For this reason, the following alternative based on the fibered barcode is sometimes used in practice. It is known to be efficiently computable when $n = 2$.

**Definition 9.16.** *The* matching distance *between p.f.d.* $\mathbb{R}^n$*-persistence modules* $\mathbb{U}, \mathbb{V}$ *is*

$$d_{\mathrm{match}}(\mathbb{U}, \mathbb{V}) := \sup_L \left\{ d_B\big(\mathcal{B}(\mathbb{U}_L),\ \mathcal{B}(\mathbb{V}_L)\big) \right\},$$

*where the supremum is taken over all lines* $L : t \mapsto ta + b$ *with* $a \succeq \mathbb{1}$, $b \in \mathbb{R}^n$*.*

**Theorem 9.17** (see [3]). *For any two p.f.d.* $\mathbb{R}^n$*-persistence modules,*

$$d_{\mathrm{match}}(\mathbb{U}, \mathbb{V}) \leqslant d_I(\mathbb{U}, \mathbb{V}).$$

**Exercise 9.18.** *Prove that* $d_B\big(\mathcal{B}(\mathbb{U}_L),\ \mathcal{B}(\mathbb{V}_L)\big) \leqslant d_I(\mathbb{U}, \mathbb{V})$ *for all lines* $L : t \mapsto t\mathbb{1} + b$*.*

### 9.3.1 Robustness of some bipersistence modules

Finally, we come back to the primary reason for studying multiparameter persistence: achieving robustness w.r.t. outliers in the data. To state formal guarantees of this form, we need to think about point-cloud data in a more probablistic way. For a finite (multi)set $X \subseteq \mathbb{R}^n$, we write $\mu_X$ for the uniform probability measure on $X$, meaning the measure that assigns probability $1/|X|$ to each $x \in X$. The following can be thought of as a 'probabilistic Hausdorff distance' between point clouds.

**Definition 9.19.** *Let* $X, Y \subseteq \mathbb{R}^n$ *finite. The* Prohorov distance[1] *between* $\mu_X, \mu_Y$ *is*

$$d_{\mathrm{Pr}}(\mu_X, \mu_Y) := \sup_A \inf\{\delta \geqslant 0 : \mu_X(A) \leqslant \mu_Y(A^\delta) + \delta \text{ and } \mu_Y(A) \leqslant \mu_X(A^\delta) + \delta\},$$

*where* $A$ *ranges over all closed subsets of* $\mathbb{R}^n$ *and* $A^\delta := \{y \in \mathbb{R}^n : \mathrm{dist}(y, A) \leqslant \delta\}$*.*

[1]The Prohorov distance can be defined between any two measures $\mu, \nu$ on (the same) metric space $M$.

While the definition of the Prohorov distance appears a bit complicated, the following observation suggests that it captures 'robustness to outliers' in a meaningful way.

**Exercise 9.20.** *Let* $X \subsetneq Y \subseteq \mathbb{R}^n$*, finite, non-empty sets. Show that*

$$d_{Pr}(\mu_X, \mu_Y) \leqslant \frac{|Y \setminus X|}{|X|}.$$

As an example, the *normalized* multicover persistence module turns out to be stable w.r.t. Prohorov distance (i.e., it is robust to outliers).

**Definition 9.21.** *Let* $X \subseteq \mathbb{R}^n$ *be a finite point cloud. For* $r, \rho \in \mathbb{R}$*, the sets*

$$\mathcal{NMC}_\rho^r(X) := \{x \in \mathbb{R}^n : |B(x,r) \cap X| \geqslant \rho|X|\} = \{x \in \mathbb{R}^n : \mu_X\big(B(x,r) \cap X\big) \geqslant \rho\}.$$

*form a bilfiltration over* $\mathbb{R} \times \mathbb{R}^{op}$ *called the* normalized multicover bifiltration*.*

**Theorem 9.22** (see [1]). *Let* $X, Y \subseteq \mathbb{R}^n$ *finite. For all* $k \geqslant 0$*, we have*

$$d_I\left(H_k(\mathcal{NMC}(X)),\ H_k(\mathcal{NMC}(Y))\right) \leqslant d_{Pr}(\mu_X, \mu_Y).$$

*Proof.* Let $\epsilon = d_{Pr}(\mu_X, \mu_Y)$. We will show that

$$\mathcal{NMC}_\rho^r(X) \subseteq \mathcal{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y) \subseteq \mathcal{NMC}_{\rho-2\epsilon}^{r+2\epsilon}(X) \quad \forall r, \rho \in \mathbb{R}.$$

These inclusions induce the maps

$$\begin{aligned} \varphi_{r,\rho} &: H_k(\mathcal{NMC}_\rho^r(X)) \to H_k(\mathcal{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)), \\ \psi_{r+\epsilon,\rho-\epsilon} &: H_k(\mathcal{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)) \to H_k(\mathcal{NMC}_{\rho-2\epsilon}^{r+2\epsilon}(X)) \end{aligned}$$

required to show that $H_k(\mathcal{NMC}(X))$ and $H_k(\mathcal{NMC}(Y))$ are $\epsilon$-interleaved.

So, let $r, \rho \in \mathbb{R}$, and suppose that $x \in \mathcal{NMC}_\rho^r(X)$. By definition, this means that $\mu_X(B(x,r) \cap X) \geqslant \rho$. Using the definition of the Prohorov distance, and the triangle-inequality, this implies that

$$\rho \leqslant \mu_X\big(B(x,r) \cap X\big) \leqslant \mu_Y\big(B(x,r)^\epsilon \cap Y\big) + \epsilon \leqslant \mu_Y\big(B(x, r+\epsilon) \cap Y\big) + \epsilon.$$

That is to say, $\mu_Y(B(x, r+\epsilon) \cap Y) \geqslant \rho - \epsilon$, meaning $x \in \mathcal{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)$, giving us the first inclusion. The second inclusion follows from an analogous argument, switching the roles of $X$ and $Y$. □

There are many similar theorems for other bipersistence modules, including the degree-Rips bifiltration. They are often somewhat hard to state and prove, relying on variants of the Prohorov distance, as well as variants of the interleaving distance [1].

## Questions

35. *How can we define persistence modules over multiple parameters?* Discuss the technical definitions that are needed.
36. *What are some common bipersistence modules? How do they relate to the (1-parameter) persistence modules of earlier chapters?*
37. *What is a major upside of multiparameter persistence? What is a major downside?* Discuss robustness and representability.
38. *How can we visualize multiparameter persistence modules?*

## References

[1] Andrew J. Blumberg and Michael Lesnick, Stability of 2-parameter persistent homology. *Found. Comput. Math.*, 24/2, (2024), 385–427.

[2] Magnus Bakke Botnan and Michael Lesnick, An Introduction to Multiparameter Persistence. 2023.

[3] Claudia Landi, The rank invariant stability via interleavings. In *Research in computational topology*, vol. 13 of *Assoc. Women Math. Ser.*, pp. 1–10, Springer, Cham, 2018.