

Chapter 9

Multiparameter persistence

This chapter closely follows the introductory paper [2] on multiparameter persistence. The interested reader is referred there for a (much) more comprehensive overview of the topic.

As we have seen, the persistence modules arising from the Čech or Vietoris-Rips complexes are *stable* under small perturbations of the underlying data, but not *robust*. That is, even a small number of outliers can drastically change the persistence diagram or barcode. We could try to remedy this problem by making these complexes *density-aware* in some way. For the Vietoris-Rips complex, a typical way to do this is as follows. For a vertex v in a simplicial complex, its degree $\deg(v)$ is the number of edges (1-simplices) in the complex that contain v . Then, for $d \in \mathbb{N}$, the degree- d Vietoris-Rips complex is

$$\mathbb{VR}_d^r(X) := \{\sigma \in \mathbb{VR}^r(X) : \text{each vertex of } \sigma \text{ has degree at least } d\}.$$

Note that vertices corresponding to data points in high-density areas of X will have relatively higher degree, whereas outliers will have relatively lower degree. Thus, for d large enough, we should expect this modification to reduce the impact of outliers. On the other hand, if d is too large, we are ‘throwing away the baby with the bathwater’. So, the question is: how to choose d ? Here, we run into the same issue that originally motivated persistence: there might not be one choice of d that accurately reflects the entire data set. Even if this choice would exist, it might be hard to determine. Instead, we would like to consider all choices of d simultaneously. The solution is to look at persistence w.r.t. both the *scale parameter* r and the *density parameter* d at the same time. In this chapter, we formalize such *multiparameter persistence*, and take a look at the representability and robustness of the resulting *multiparameter persistence modules*.

9.1 Persistence modules indexed by a poset

A persistence module (indexed by $\mathbb{R}$) consists of a family of vector spaces U_a , $a \in \mathbb{R}$, together with commuting maps $U_a \rightarrow U_b$ for $a \leq b$. If we want to define persistence

modules indexed by more exotic sets, we need to first extend the meaning of ‘ $\leq$ ’.

Definition 9.1. A poset (*partially ordered set*) is a tuple $(P, \preceq)$ consisting of a set P and a binary relation $\preceq$ on P satisfying

- $a \preceq a$ for all $a \in P$;
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in P$;
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in P$;

We will sometimes simply write P if the (partial) ordering $\preceq$ is clear from context.

Note that partial orderings allow for elements to be *incomparable*, i.e., it can happen for $a, b \in P$ that neither $a \preceq b$ nor $b \preceq a$ holds. In contrast, under a *total ordering* all elements in P must be comparable. All the posets we will see in this chapter are constructed from the following examples:

- The reals $\mathbb{R} = (\mathbb{R}, \leq)$ and natural numbers $\mathbb{N} = (\mathbb{N}, \leq)$ with their usual ordering are posets.
- If $(P, \preceq)$ is a poset, and $Q \subseteq P$, then $(Q, \preceq)$ is a poset;
- For any poset $(P, \preceq)$, we have the *opposite poset* $P^{\text{op}} = (P, \succeq)$, where

$$a \succeq b \iff b \preceq a \quad \text{for all } a, b \in P.$$

- For two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$ we have the *product poset* $(P \times Q, \preceq)$, where

$$(p, q) \preceq (p', q') \iff p \preceq_P p' \text{ and } q \preceq_Q q' \quad \text{for all } p, p' \in P \text{ and } q, q' \in Q.$$

We can now define filtrations and persistence modules indexed by an arbitrary poset, generalizing our definitions from earlier chapters.

Definition 9.2. Let $(P, \preceq)$ be a poset. A family $(X_p)_{p \in P}$ of topological spaces (or simplicial complexes) is called a *P-filtration* if $X_p \subseteq X_{p'}$ for all $p, p' \in P$ with $p \preceq p'$.

Definition 9.3. Let $(P, \preceq)$ be a poset, and let $\mathbb{F}$ be a field. A *P-persistence module* (over $\mathbb{F}$) consists of

- An $\mathbb{F}$ -vector space U_p for each $p \in P$;
- A linear map $u_{p,p'} : U_p \rightarrow U_{p'}$ for each $p, p' \in P$ with $p \preceq p'$, satisfying $u_{p_2,p_3} \circ u_{p_1,p_2} = u_{p_1,p_3}$ for all $p_1, p_2, p_3 \in P$ with $p_1 \preceq p_2 \preceq p_3$.

Observation 9.4. If $(X_p)_{p \in P}$ is a *P-filtration* for some poset P , then taking k -dimensional homology gives us a *P-persistence module* $H_k(X_p)$ (where, as before, the maps $u_{p,p'} : H_k(X_p) \rightarrow H_k(X_{p'})$ are induced by the inclusions $X_p \hookrightarrow X_{p'}$).

We can now give two examples of filtrations indexed by $P = \mathbb{R} \times \mathbb{N}^{\text{op}}$, each of which induces a P -persistence module via the observation above. The first one formalizes the discussion at the beginning of the chapter. The second one applies a similar idea to the Čech complex.

Example 9.5. *Let X be a finite metric space. Then the family $(\mathbb{V}\mathbb{R}_d^r(X))_{r \in \mathbb{R}, d \in \mathbb{N}}$ defined above is a filtration indexed by $\mathbb{R} \times \mathbb{N}^{\text{op}}$. It is called the degree-Rips bifiltration.*

Example 9.6. *Let $X \subseteq \mathbb{R}^n$ be a finite point cloud. For $r \in \mathbb{R}$ and $d \in \mathbb{N}$, define*

$$\mathcal{MC}_d^r = \{x \in \mathbb{R}^n : |B(x, r) \cap X| \geq d\} \subseteq \mathbb{R}^n.$$

Note that, for $d = 1$, this is just the union-of-balls used to define the Čech complex. The family $(\mathcal{MC}_d^r(X))_{r \in \mathbb{R}, d \in \mathbb{N}}$ is a filtration (of topological spaces) indexed by $\mathbb{R} \times \mathbb{N}^{\text{op}}$. It is called the multicover bifiltration.

Both of the filtrations above can also be thought of as indexed over $\mathbb{R}^2$ (after a reparameterization). In general, we call an $\mathbb{R}^n$ -filtration (resp. $\mathbb{R}^n$ -persistence module) an n -parameter filtration (resp. n -parameter persistence module). For $n \geq 2$, we also say ‘multiparameter’. For $n = 2$ the terms bifiltration and bipersistence module are common.

9.2 Representing persistence modules indexed by posets

9.2.1 Barcodes?

Recall that a p.f.d. persistence module $\mathbb{U}$ indexed by $\mathbb{R}$ can be uniquely represented by a barcode. In Chapter 6, we saw that this follows from the fact that any such $\mathbb{U}$ can be decomposed into interval modules in a unique way:

$$\mathbb{U} \cong \bigoplus_{i \in I} \mathbb{I}_{\langle a, b \rangle}.$$

Each interval in this decomposition corresponds to a bar in the barcode of $\mathbb{U}$. We could hope a similar statement holds for modules indexed by any poset P . The following definition and theorem should give us some hope:

Definition 9.7. *A persistence module $\mathbb{U}$ is called indecomposable if $\mathbb{U} \cong \mathbb{U}_1 \oplus \mathbb{U}_2$ implies that $\mathbb{U}_1 = 0$ or $\mathbb{U}_2 = 0$.*

Exercise 9.8. *Show that interval modules (indexed by $\mathbb{R}$) are indecomposable.*

Theorem 9.9. *Let $\mathbb{U}$ be a p.f.d. persistence module (indexed by a poset P). Then, there is a unique decomposition:*

$$\mathbb{U} \cong \bigoplus_{i \in I} \mathbb{U}_i,$$

where each persistence module $\mathbb{U}_i$ is indecomposable.

Understanding P -persistence modules thus boils down to understanding indecomposables (indexed by P). For $P = \mathbb{R}$, it turns out that indecomposables are precisely interval modules, leading to Theorem 6.36. However, for general P , the situation is much more complicated. Without going into details: It is not possible to parameterize indecomposables in terms of some nice family of subsets of P . The decomposition of Theorem 9.9 therefore does not lead to a workable notion of barcode in general. In fact, it is not possible to define any reasonable notion of a barcode for P -persistence modules, in the following sense.

Definition 9.10. Let $(P, \preceq)$ be a poset and let $\mathbb{U}$ be a P -persistence module. We say a multiset $\mathcal{B}$ of subsets of P is a *reasonable barcode* for $\mathbb{U}$ if

$$\text{rank}(u_{p,p'}) = |\{B \in \mathcal{B} : p, p' \in B\}| \quad (\forall p \preceq p').$$

That is, the rank of the map $u_{p,p'} : \mathbb{U}_p \rightarrow \mathbb{U}_{p'}$ can be computed by counting the number of ‘bars’ that contain both p and p' .

Exercise 9.11. Show that the usual barcode for a p.f.d. persistence module indexed by $\mathbb{R}$ is reasonable in the above sense.

Exercise 9.12. For $p \in P$, show that $\dim \mathbb{U}_p$ is greater than or equal to the number of bars that contain p in a reasonable barcode for $\mathbb{U}$.

Example 9.13. Let $P = \{0, 1, 2\} \times \{0, 1, 2\}$ and consider the following persistence module indexed by P :

$$\mathbb{U} = \begin{array}{ccccc} \mathbb{F} & \xrightarrow{\text{id}} & \mathbb{F} & \longrightarrow & 0 \\ \text{id} \uparrow & & g \uparrow & & \uparrow \\ \mathbb{F} & \xrightarrow{f} & \mathbb{F}^2 & \xrightarrow{h} & \mathbb{F} \\ \uparrow & & j \uparrow & & \text{id} \uparrow \\ 0 & \longrightarrow & \mathbb{F} & \xrightarrow{\text{id}} & \mathbb{F} \end{array}, \quad \text{where} \quad \begin{aligned} f &: a \mapsto (a, 0) \\ g &: (a, b) \mapsto a \\ h &: (a, b) \mapsto a + b \\ j &: a \mapsto (0, a) \end{aligned}$$

We claim $\mathbb{U}$ cannot have a reasonable barcode. To see this, suppose $\mathcal{B}$ is a reasonable barcode for $\mathbb{U}$. Note that

$$\text{rank}(h \circ f) = \text{rank}(g \circ f) = \text{rank}(h \circ j) = 1.$$

By the reasonability assumption, there thus must be subsets $I, J, K \in \mathcal{B}$ with

$$(0, 1), (2, 1) \in I, \quad (0, 1), (1, 2) \in J, \quad (1, 0), (2, 1) \in K.$$

Since $\dim \mathbb{U}_{0,1} = \dim \mathbb{U}_{2,1} = 1$, we know by Exercise 9.12 that $(0, 1)$ and $(2, 1)$ occur in at most one element of $\mathcal{B}$. But that means that $I = J$, and $I = K$, and so in fact $I = J = K \supseteq \{(0, 1), (2, 1), (1, 2)\}$. Thus, using reasonability again, we find that

$$\text{rank}(g \circ j) \geq 1,$$

contradicting the fact that $g \circ j = 0$.

For completeness, we note that it is possible to have reasonable barcodes for multiparameter persistence modules if we allow ‘bars’ to occur with *negative* multiplicity. This leads to so-called *signed barcodes*. Furthermore, there are several special classes of multiparameter persistence modules which do admit an interval decomposition (note that we did not formally define what an interval is in the poset-setting!). This is true in particular for so-called *interlevel persistence*. We will not discuss this further here.

9.2.2 Other representations and visualizations

Even though we cannot hope for a barcode-like representation of multiparameter persistence modules, there are still several useful (partial) representations available. We mention just three of them here:

- The *Hilbert-function* of a p.f.d. persistence module $\mathbb{U}$ indexed by P sends $p \in P$ to $\dim U_p \in \mathbb{N}$. It can be thought of as a ‘homological heatmap’.
- The *rank-invariant* of $\mathbb{U}$ sends a pair (p, p') , $p \preceq p'$ to the rank $\text{rank}(u_{p,p'})$ of the map between U_p and $U_{p'}$. Note that any two modules with $\mathbb{U} \cong \mathbb{V}$ have the same rank invariant. For modules indexed by $\mathbb{R}$, this implication also holds in the other direction. However, for general P , there exist nonisomorphic modules with the same rank-invariant;
- Let $L : t \mapsto ta + b$ be an affine line in $\mathbb{R}^2$ with non-negative slope. Then the restriction $\mathbb{U}_L$ of a persistence module $\mathbb{U}$ indexed by $\mathbb{R}^2$ to the line L can be viewed as a persistence module indexed by $\mathbb{R}$ (with its usual, total ordering). The *Fibered Barcode* of $\mathbb{U}$ sends each such line L to the barcode $\mathcal{B}(\mathbb{U}_L)$. This idea can be extended to lines in $\mathbb{R}^n$;

Exercise 9.14. *Show that the rank-invariant of an $\mathbb{R}^2$ -persistence module can be recovered from its fibered barcode, and vice versa.*

The RIVET software package (rivet.readthedocs.io) allows you to compute and visualize each of the invariants above for (among others) the degree-Rips bifiltration.

9.3 Distances and robustness for P -persistence modules

As we generally do not have barcodes for P -persistence modules, we cannot rely on the Bottleneck distance to compare them. Instead, we work directly with interleaving distance, which is still well-defined. For simplicity, we only consider $P = \mathbb{R}^n$.

Definition 9.15. *Write $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$. Two $\mathbb{R}^n$ -persistence modules $\mathbb{U}, \mathbb{V}$ are ϵ -interleaved if there exist two families of linear maps, $\varphi_a : U_a \rightarrow V_{a+\epsilon\mathbf{1}}$ and*

$\psi_a : V_a \rightarrow U_{a+\epsilon \mathbb{1}}$, $a \in \mathbb{R}^n$, such that the following diagrams (and their symmetric counterparts obtained by exchanging the role of U and V) are commutative:

$$\begin{array}{ccc}
 U_a & \xrightarrow{u_{a,a'}} & U_{a'} \\
 \searrow \varphi_a & & \searrow \varphi_{a'} \\
 & V_{a+\epsilon \mathbb{1}} & \xrightarrow{v_{a+\epsilon \mathbb{1}, a'+\epsilon \mathbb{1}}} V_{a'+\epsilon \mathbb{1}}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 U_a & \xrightarrow{u_{a,a+2\epsilon \mathbb{1}}} & U_{a+2\epsilon \mathbb{1}} \\
 \searrow \varphi_a & & \nearrow \psi_{a+\epsilon \mathbb{1}} \\
 & V_{a+\epsilon \mathbb{1}} &
 \end{array}$$

The interleaving distance between U, V is $d_I(U, V) := \inf_{\epsilon \geq 0} \{U, V \text{ are } \epsilon\text{-interleaved}\}$.

The definition above agrees with our earlier definition of interleaving when $n = 1$. Contrary to the 1-dimensional case, computing the interleaving distance for $n \geq 2$ is an NP-hard problem (even if the persistence modules U, V are given to us in a ‘nice’ form). For this reason, the following alternative based on the fibered barcode is sometimes used in practice. It is known to be efficiently computable when $n = 2$.

Definition 9.16. The matching distance between p.f.d. $\mathbb{R}^n$ -persistence modules U, V is

$$d_{\text{match}}(U, V) := \sup_L \left\{ d_B(\mathcal{B}(U_L), \mathcal{B}(V_L)) \right\},$$

where the supremum is taken over all lines $L : t \mapsto ta + b$ with $a \succeq \mathbb{1}$, $b \in \mathbb{R}^n$.

Theorem 9.17 (see [3]). For any two p.f.d. $\mathbb{R}^n$ -persistence modules,

$$d_{\text{match}}(U, V) \leq d_I(U, V).$$

Exercise 9.18. Prove that $d_B(\mathcal{B}(U_L), \mathcal{B}(V_L)) \leq d_I(U, V)$ for all lines $L : t \mapsto t\mathbb{1} + b$.

9.3.1 Robustness of some bipersistence modules

Finally, we come back to the primary reason for studying multiparameter persistence: achieving robustness w.r.t. outliers in the data. To state formal guarantees of this form, we need to think about point-cloud data in a more probabilistic way. For a finite (multi)set $X \subseteq \mathbb{R}^n$, we write μ_X for the uniform probability measure on X , meaning the measure that assigns probability $1/|X|$ to each $x \in X$. The following can be thought of as a ‘probabilistic Hausdorff distance’ between point clouds.

Definition 9.19. Let $X, Y \subseteq \mathbb{R}^n$ finite. The Prohorov distance¹ between μ_X, μ_Y is

$$d_{\text{Pr}}(\mu_X, \mu_Y) := \sup_A \inf \{ \delta \geq 0 : \mu_X(A) \leq \mu_Y(A^\delta) + \delta \text{ and } \mu_Y(A) \leq \mu_X(A^\delta) + \delta \},$$

where A ranges over all closed subsets of $\mathbb{R}^n$ and $A^\delta := \{y \in \mathbb{R}^n : \text{dist}(y, A) \leq \delta\}$.

¹The Prohorov distance can be defined between any two measures μ, ν on (the same) metric space M .

While the definition of the Prohorov distance appears a bit complicated, the following observation suggests that it captures ‘robustness to outliers’ in a meaningful way.

Exercise 9.20. Let $X \subsetneq Y \subseteq \mathbb{R}^n$, finite, non-empty sets. Show that

$$d_{\text{Pr}}(\mu_X, \mu_Y) \leq \frac{|Y \setminus X|}{|X|}.$$

As an example, the *normalized* multicover persistence module turns out to be stable w.r.t. Prohorov distance (i.e., it is robust to outliers).

Definition 9.21. Let $X \subseteq \mathbb{R}^n$ be a finite point cloud. For $r, \rho \in \mathbb{R}$, the sets

$$\text{NMC}_\rho^r(X) := \{x \in \mathbb{R}^n : |B(x, r) \cap X| \geq \rho|X|\} = \{x \in \mathbb{R}^n : \mu_X(B(x, r) \cap X) \geq \rho\}.$$

form a bifiltration over $\mathbb{R} \times \mathbb{R}^{\text{op}}$ called the *normalized multicover bifiltration*.

Theorem 9.22 (see [1]). Let $X, Y \subseteq \mathbb{R}^n$ finite. For all $k \geq 0$, we have

$$d_I(H_k(\text{NMC}(X)), H_k(\text{NMC}(Y))) \leq d_{\text{Pr}}(\mu_X, \mu_Y).$$

Proof. Let $\epsilon = d_{\text{Pr}}(\mu_X, \mu_Y)$. We will show that

$$\text{NMC}_\rho^r(X) \subseteq \text{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y) \subseteq \text{NMC}_{\rho-2\epsilon}^{r+2\epsilon}(X) \quad \forall r, \rho \in \mathbb{R}.$$

These inclusions induce the maps

$$\begin{aligned} \varphi_{r,\rho} &: H_k(\text{NMC}_\rho^r(X)) \rightarrow H_k(\text{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)), \\ \psi_{r+\epsilon,\rho-\epsilon} &: H_k(\text{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)) \rightarrow H_k(\text{NMC}_{\rho-2\epsilon}^{r+2\epsilon}(X)) \end{aligned}$$

required to show that $H_k(\text{NMC}(X))$ and $H_k(\text{NMC}(Y))$ are ϵ -interleaved.

So, let $r, \rho \in \mathbb{R}$, and suppose that $x \in \text{NMC}_\rho^r(X)$. By definition, this means that $\mu_X(B(x, r) \cap X) \geq \rho$. Using the definition of the Prohorov distance, and the triangle-inequality, this implies that

$$\rho \leq \mu_X(B(x, r) \cap X) \leq \mu_Y(B(x, r) \cap Y) + \epsilon \leq \mu_Y(B(x, r + \epsilon) \cap Y) + \epsilon.$$

That is to say, $\mu_Y(B(x, r + \epsilon) \cap Y) \geq \rho - \epsilon$, meaning $x \in \text{NMC}_{\rho-\epsilon}^{r+\epsilon}(Y)$, giving us the first inclusion. The second inclusion follows from an analogous argument, switching the roles of X and Y . $\square$

There are many similar theorems for other bipersistence modules, including the degree-Rips bifiltration. They are often somewhat hard to state and prove, relying on variants of the Prohorov distance, as well as variants of the interleaving distance [1].

Questions

35. *How can we define persistence modules over multiple parameters?* Discuss the technical definitions that are needed.
36. *What are some common bipersistence modules? How do they relate to the (1-parameter) persistence modules of earlier chapters?*
37. *What is a major upside of multiparameter persistence? What is a major downside?* Discuss robustness and representability.
38. *How can we visualize multiparameter persistence modules?*

References

- [1] Andrew J. Blumberg and Michael Lesnick, Stability of 2-parameter persistent homology. *Found. Comput. Math.*, 24/2, (2024), 385–427.
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- [3] Claudia Landi, The rank invariant stability via interleavings. In *Research in computational topology*, vol. 13 of *Assoc. Women Math. Ser.*, pp. 1–10, Springer, Cham, 2018.