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Theoretische Informatik (Kernfach) *SS 2005*
Exercise Set 10

Exercise 1

Suppose that a polynomial $p(x_1, \ldots, x_n, y)$ in $n+1$ variables is given by a black box that, given concrete values for $x_1, \ldots, x_n$ and y, returns the value of the polynomial. Let m be the maximum degree of y in p and write $p(x_1, \ldots, x_n, y) = \sum_{i=0}^{m} y^i p_i(x_1, \ldots, x_n)$. Given an integer k and numbers $r_1, \ldots, r_n$, how can we compute $p_k(r_1, \ldots, r_n)$ (using only the black box)? How can we test whether $p_k(x_1, \ldots, x_n)$ is a nonzero polynomial, assuming that $p_k(x_1, \ldots, x_n)$ has degree d? For simplicity, assume that everything happens over the rationals.

Exercise 2

Consider a bipartite graph in which the edges are colored red and blue. Extend the randomized algorithm discussed in class to handle the following problem: Given such a colored bipartite graph and an integer k, is there a perfect matching that contains exactly k red edges?

Hint: Use Exercise 1.

(*Remark:* No polynomial-time deterministic algorithm for this problem seems to be known.)

Exercise 3

(a) Transform the CNF formula $(x_1 \vee \bar{x}_2 \vee x_3 \vee x_5) \wedge (\bar{x}_1 \vee x_4 \vee x_5 \vee \bar{x}_6 \vee \bar{x}_7)$ into a 3CNF formula by introducing auxiliary variables (as described in the lecture notes).

(b) Formulate a general construction that transforms any CNF formula ϕ into a 3CNF formula ψ. Prove that ψ is satisfiable if and only if ϕ is satisfiable.
What is the number of clauses used in the new formula ψ?

Exercise 4

Let ψ be a proper 3CNF formula (no variable occurs more than once per clause) with m clauses and let X be the number of clauses in ψ satisfied by a random assignment.

(a) Show the existence of an assignment for ψ that satisfies a fraction of $\frac{7}{8}$ of all clauses by showing that $E[X] = \frac{7}{8}m$.

(b) Let $\alpha_k \in \{0, 1\}$ denote an assignment to the variable x_k and let $n_j(\alpha_1, \ldots, \alpha_i)$ denote the number of clauses in ψ that
(i) contain the literal x_{i+1} ($\bar{x}_{i+1}$ does not count!),
(ii) are not satisfied by the partial assignment[1] $\alpha_1, \ldots, \alpha_i$, and
(iii) contain exactly j yet unassigned variables ($0 \leq j \leq 3$)
and similarly $\bar{n}_j(\alpha_1, \ldots, \alpha_i)$ is defined as the number of clauses that contain the literal $\bar{x}_{i+1}$, are not satisfied by $\alpha_1, \ldots, \alpha_i$ and have j unassigned variables.

$$\begin{aligned} \text{Let}\quad w(\alpha_1, \ldots, \alpha_i) &= n_3(\alpha_1, \ldots, \alpha_i) + 2n_2(\alpha_1, \ldots, \alpha_i) + 4n_1(\alpha_1, \ldots, \alpha_i) - \bar{n}_3(\alpha_1, \ldots, \alpha_i) \\ &\quad -2\bar{n}_2(\alpha_1, \ldots, \alpha_i) - 4\bar{n}_1(\alpha_1, \ldots, \alpha_i). \end{aligned}$$

We define $E_{\alpha_1, \ldots, \alpha_i}[X]$ to be the expected number of clauses satisfied when the first i variables are assigned according to the assignment $\alpha_1, \ldots, \alpha_i$ and the remaining variables are assigned at random.

[1] A clause is satisfied as soon as one of its literals is set to true – even if not all literals have been assigned a value.

Note that $E_{\alpha_1,\ldots,\alpha_n}[X]$ is just the number of clauses satisfied by the assignment $\alpha_1, \ldots, \alpha_n$, thus there is no randomness. If no variables are preassigned a fixed value then $E[X]$ (in correspondence with our notation) is just the expected number of clauses satisfied by a random assignment as in (a).

Prove that

$$E_{\alpha_1,\ldots,\alpha_i,1}[X] = E_{\alpha_1,\ldots,\alpha_i}[X] + \frac{w(\alpha_1, \ldots, \alpha_i)}{8},$$

and that

$$E_{\alpha_1,\ldots,\alpha_i,0}[X] = E_{\alpha_1,\ldots,\alpha_i}[X] - \frac{w(\alpha_1, \ldots, \alpha_i)}{8}.$$

(c) Show that the following *deterministic* polynomial-time algorithm has approximation ratio $\frac{7}{8}$. (The algorithm is a derandomization of the probabilistic argument from (a).)

Assign 0/1 values one by one to the variables occurring in ψ resulting in an assignment $\alpha_1^*, \ldots, \alpha_n^*$ (thus, we first fix the value α_1^*, then the value α_2^*, $\ldots$). When assigning a 0/1 value to α_{i+1}^* base your decision on the weight function $w(\alpha_1^*, \ldots, \alpha_i^*)$. That is

$$\alpha_{i+1}^* = \begin{cases} 1 & \text{if } w(\alpha_1^*, \ldots, \alpha_i^*) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

<u>Hint:</u> By (b), we have $E_{\alpha_1^*,\ldots,\alpha_{i+1}^*}[X] \geq E_{\alpha_1^*,\ldots,\alpha_i^*}[X]$, for $0 \leq i < n$.

Due-date: 23.06.05 in the lecture