Topological Methods in Combinatorics and Geometry
(Predoc Course)

Summary of exam requirements

Basic level of knowledge (required for passing the exam); includes understanding of the definitions and of the main theorems, and the ability to explain the ideas of the proof for several of the main results.

1. Geometric notions: hyperplane, halfspace, convex hull, convex polytope, its faces.

2. Topological space, open set, closed set, continuous map, homeomorphism; compact space, properties of continuous functions on compact spaces (attaining maximum, uniform continuity).

3. Homotopy of maps, homotopy equivalence of spaces, deformation retraction.

4. Affine independence, simplex. Geometric and abstract simplicial complex, geometric realization; polyhedron, triangulation. Simplicial map, its affine extension to polyhedra. Triangulation of S^n using the crosspolytope, combinatorial structure of the faces.

5. Moment curve; every (finite) d-dimensional simplicial complex is realizable in $\mathbb{R}^{2d+1}$.

6. Poset, order complex, first barycentric subdivision (combinatorial and geometric views).

7. Borsuk–Ulam theorem: versions (1.1),(1.2),(1.3),(2.1),(3.1); their equivalence.

8. Mass distribution (at least on intuitive level), ham-sandwich theorem (continuous and discrete version, general position version). Akiyama–Alon theorem, necklace theorem for two thieves.

9. Kneser's conjecture with at least one of the proofs (Bárány's or Dol'nikov's).

10. Quotient space (topology), sum, wedge. Contracting a contractible subcomplex is a homotopy equivalence.

11. Join of simplicial complexes, join of topological spaces (general definition and representation for bounded subspaces of $\mathbb{R}^d$). Join of maps. Cone, suspension.

12. k-connected space. S^n is $(n-1)$-connected but not n-connected.

13. $\mathbb{Z}_2$-space, $\mathbb{Z}_2$-action. Free $\mathbb{Z}_2$-space. $\mathbb{Z}_2$-map. Simplicial $\mathbb{Z}_2$-complex (the action is simplicial). Join of $\mathbb{Z}_2$-spaces.

14. $\mathbb{Z}_2$-index, properties (i)–(v).

15. Radon's theorem, topological Radon's theorem (formulation).

16. Deleted join of a simplicial complex K_Δ^{*2}; deleted join $(\mathbb{R}^d)_\Delta^{*2}$ (realize that $\|\mathsf{K}_\Delta^{*2}\|$ is generally different from $\|\mathsf{K}\|_\Delta^{*2}$!). Deleted join of the n-simplex is an S^n.

17. Principle of the proof of topological Radon (if a bad map existed, take its join with itself, regard it as a $\mathbb{Z}_2$-map $\|(\sigma^{d+1})_\Delta^{*2}\| \to (\mathbb{R}^d)_\Delta^{*2}$, show that such a $\mathbb{Z}_2$-map is impossible by estimating the indices).

18. Van Kampen–Flores theorem (formulation), index of the deleted join and embedding into $\mathbb{R}^d$, Sarkaria's coloring/embedding theorem (with Kneser colorings—formulation), at least one application.

Advanced level of knowledge (required for a good grade); includes the proofs of the main theorems.

1. At least one proof of the Borsuk–Ulam theorem.

2. Proofs of the three versions of ham-sandwich (continuity of the map in the continuous version can be omitted). Examples of mass partition theorems.

3. Proof of Kneser's conjecture via Gale's lemma, Schrijver's theorem, Dol'nikov's theorem.

4. Homotopy extension property and its use.

5. A simplicial complex is k-connected iff its $(k+1)$-skeleton is k-connected, connectivity of the join (without proofs).

6. Definition of CW-complex, some examples.

7. Deleted product of a space and of a simplicial complex, how topological Radon can be proved using deleted products. Computation of $\operatorname{ind}_{\mathbb{Z}_2}((\mathbb{R}^d)_\Delta^{*2})$.

8. Sarkaria's inequality, proof of Van Kampen–Flores, proof of Sarkaria's coloring/embedding theorem.

9. G-space, examples; G-map. E_nG spaces, G-index (especially for $G = \mathbb{Z}_p$). Properties of the G-index, Dold's theorem.

10. p-fold 2-wise deleted product of a simplicial complex $(\mathsf{K}_{\Delta(2)}^{*p})$, p-fold p-wise deleted join $(\mathbb{R}^d)_\Delta^{*p}$, the $\mathbb{Z}_p$-index of $(\mathbb{R}^d)_\Delta^{*p}$ is at most $(d+1)(p-1)-1$.

11. The $\mathbb{Z}_p$-index of $\mathsf{K}_{\Delta(2)}^{*p}$ and p-fold points in embedding into $\mathbb{R}^d$. Topological Tverberg theorem for prime p. Index and Kneser colorings.

12. Colored Tverberg theorem.