THE LIST COLORING CONJECTURE

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topics

- Introduction - The LCC
- Kernels and choosability
- Proof of the bipartite LCC

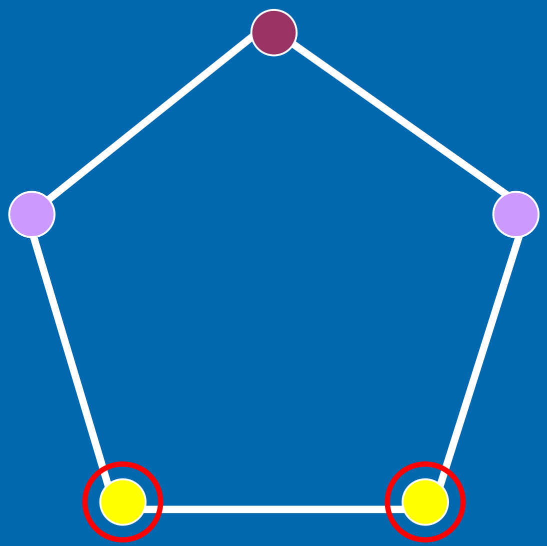
Introduction and the list coloring conjecture

vertex coloring

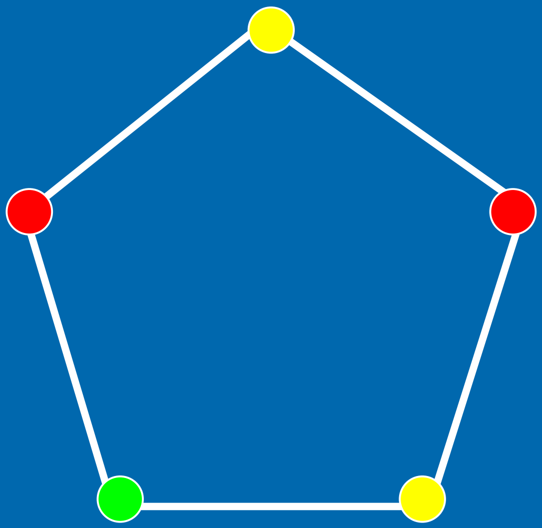
- k-coloring of a graph G: labelling f: V(G) → S with |S|=k. The labels are called colors
- A k-coloring is called proper if adjacent vertices have different colors
- G is k-colorable if it has a proper k-coloring
- Chromatic number:

  $\chi(G) := \min\{k \mid G \text{ is } k\text{-colorable}\}$

example



3-coloring



Proper 3-coloring

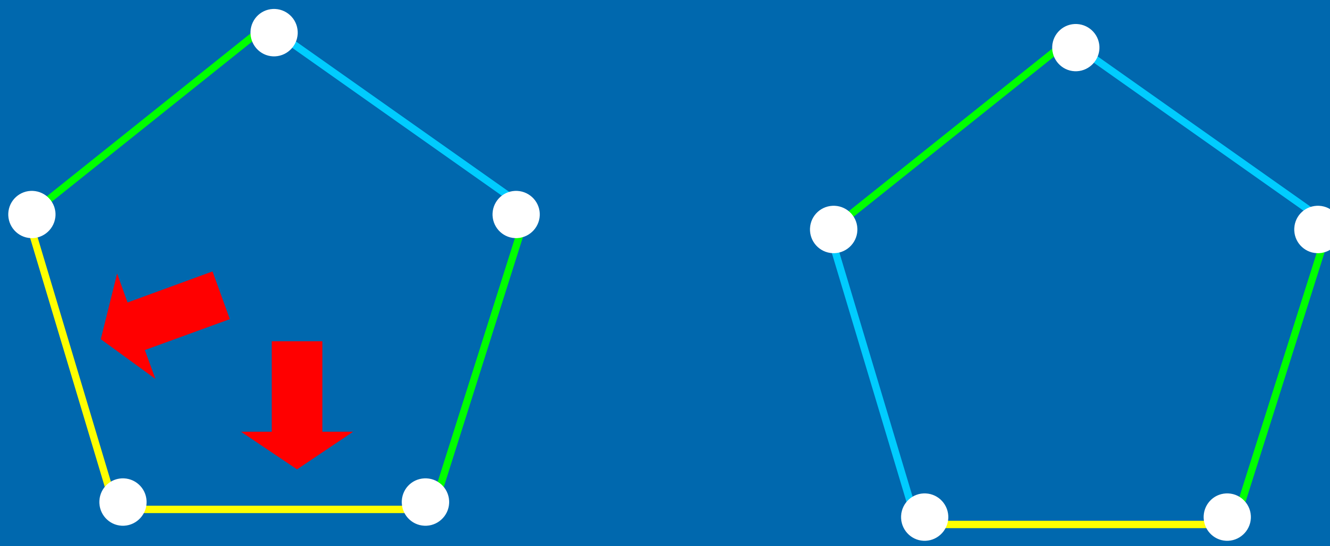
$\chi(C_5)=3$

edge coloring

- k-edge-coloring of a graph G: labelling f: E(G) → S with |S|=k.
- A k-edge-coloring is called proper if incident edges have different colors
- G is k-edge-colorable if it has a proper k-edge-coloring
- Chromatic index:

  $\chi'(G) := \min\{k \mid G \text{ is k-edge-colorable}\}$

example



3-edge-coloring

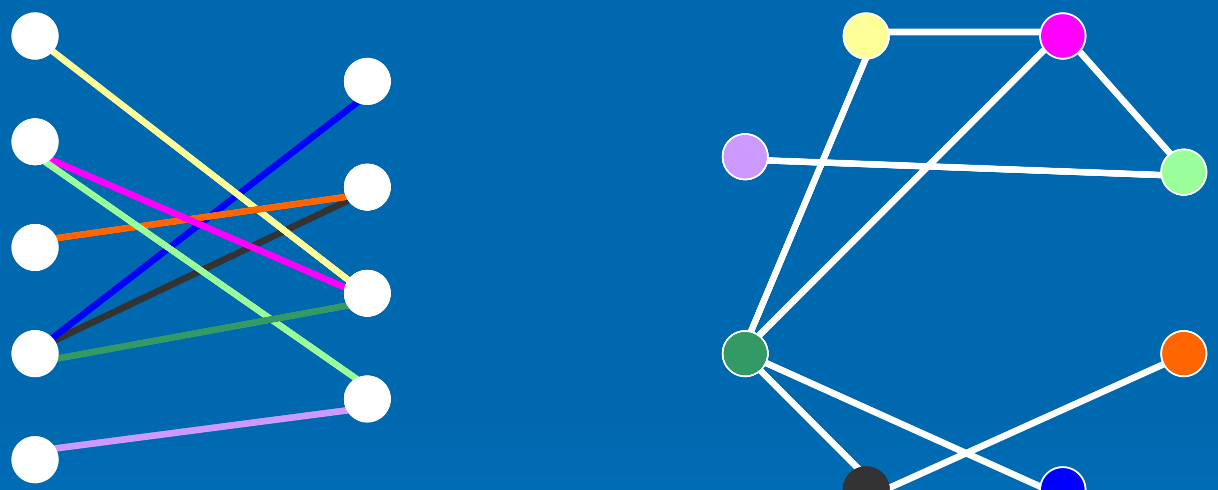
Proper 3-edge-coloring

$\chi'(C_5)=3$

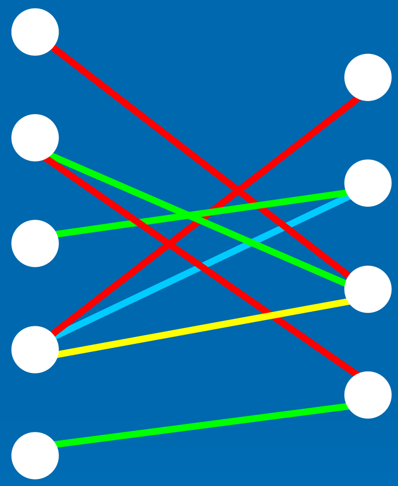
line graph

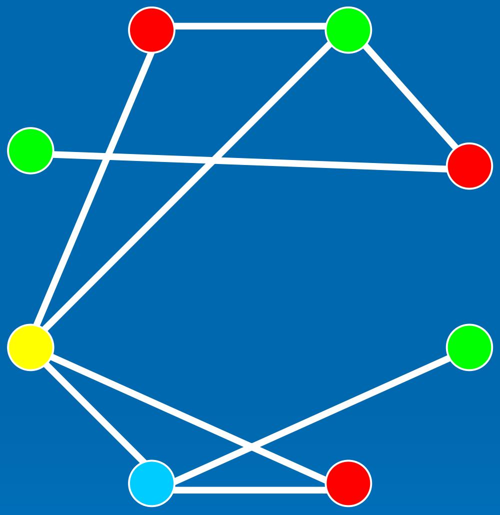
- The line graph L(G) of a graph G is the graph with vertex set V(L(G))=E(G) and edge set E(L(G))={{e,f}| e incident with f}

example



example

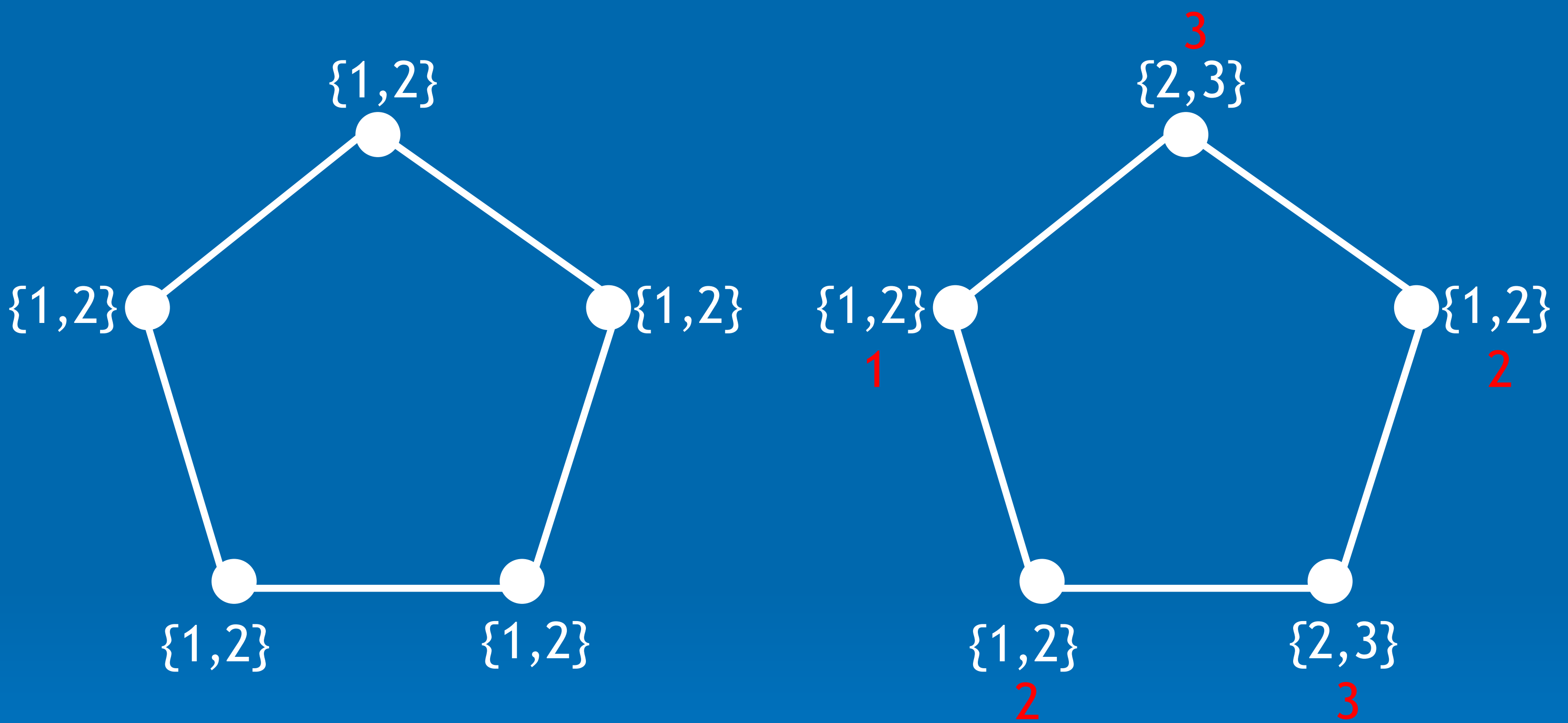




list coloring

- List coloring is a more general concept of coloring a graph
- G is n-choosable if, given any set of n colors for each vertex, we can find a proper coloring

example

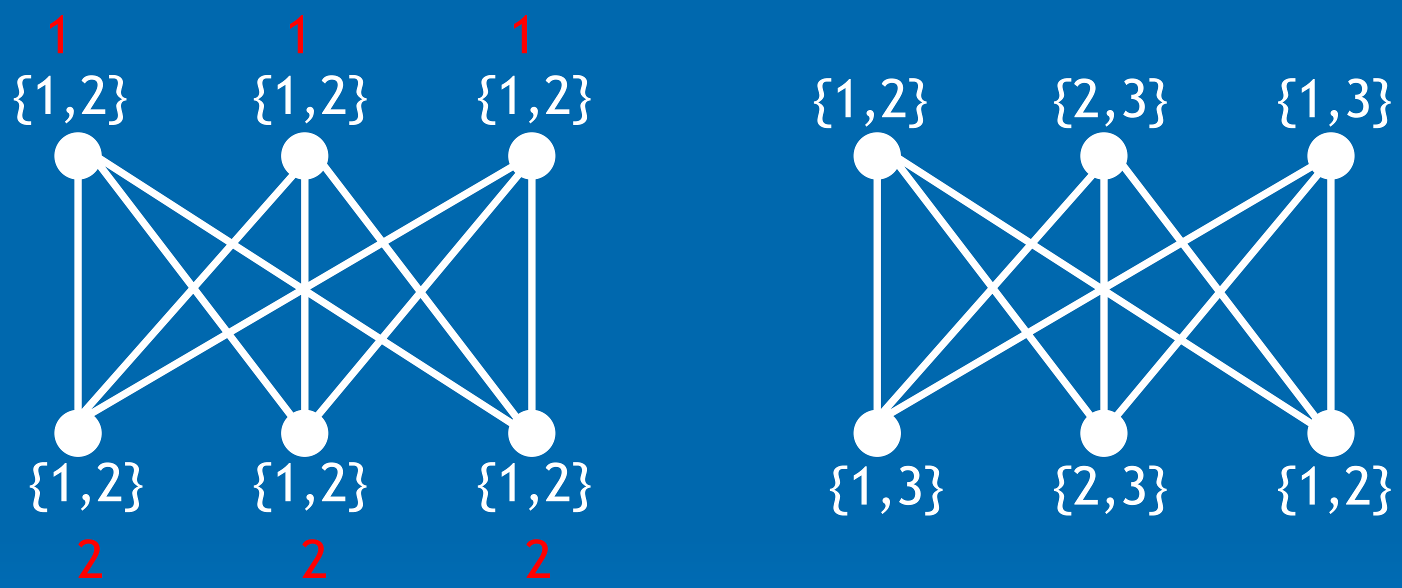


C_5 is not 2-choosable

choice number

- Choice number:
  ch(G):= min{n|G is n-choosable}
- The list chromatic index of H is the choice number of L(H)
- Clearly: ch(G) $\geq \chi$(G)

example



$\mathrm{ch}(K_{3,3}) > \chi(K_{3,3})$

list coloring conjecture

ch(G) = χ(G) whenever G is a line graph

Kernels and choosability

definitions

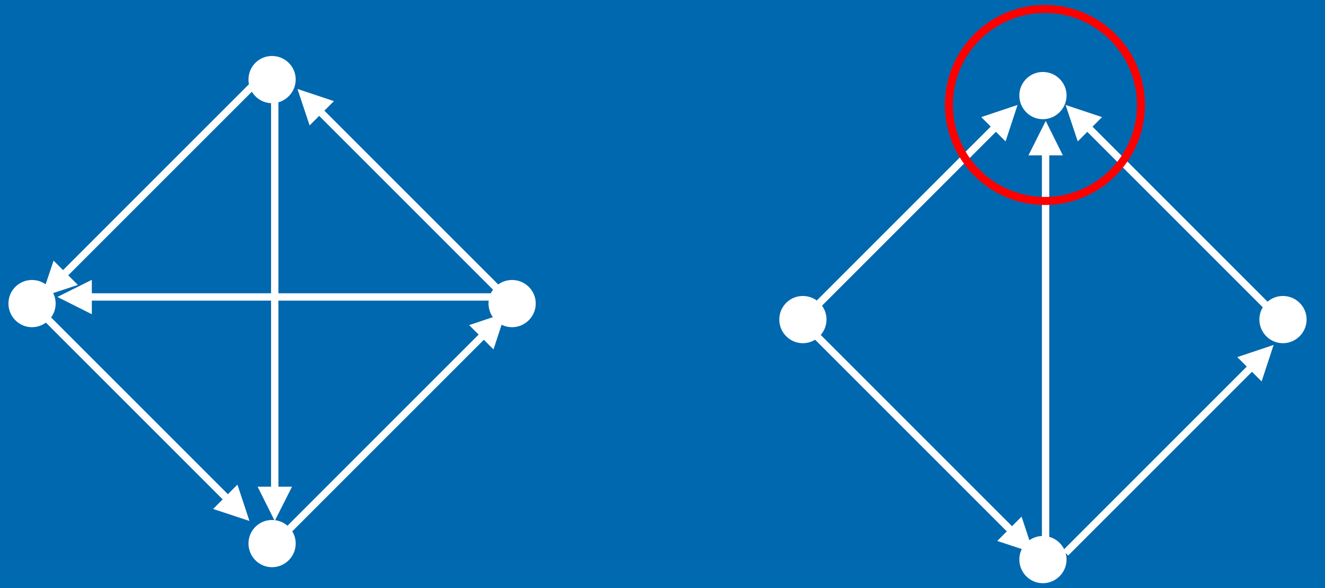
- Consider a digraph $D=(V,E)$
- Notation: $u \to v$ means that $(u,v) \in E$:

- The outdegree of v is $d^{+}(v)=|\{u|\ v \to u\}|$
- The closed neighborhood of v is $N[v] = \{u|\ v \to u \text{ or } v=u\}$
- The underlying graph of D is $G=(V,E')$ with $E' = \{\{u,v\}|\ u \to v \text{ or } v \to u\}$

kernel

- A kernel of D is an independent set $K \subseteq V$ s.t. $\forall v \in V \setminus K$ there exists an $u \in K$ with $v \rightarrow u$
- A kernel of $S \subseteq V$ is a kernel of the subdigraph induced by S

example



(f:g)-choosable

- Consider two functions f,g: $V \to \mathbb{N}$
- G is (f:g)-choosable if, given any sets A_v of colors with $f(v)=|A_v|$, we can choose subsets $B_v \subseteq A_v$ with $g(v) = |B_v|$ such that $B_u \cap B_v = \varnothing$ whenever $\{u,v\} \in E$

- Example: G is n-choosable if we take f(v)=n and g(v)=1

lemma 1

<u>Bondy, Boppana and Siegel</u>

Let D be a digraph in which

i. every induced subdigraph has a kernel

ii. f,g: V(D) → $\mathbb{N}$ are so, that $f(v) \geq \sum_{u \in N[v]} g(u)$ whenever g(v)>0

Then D is (f:g)-choosable

Proof:

- Induction on $\sum_{v \in V} g(v)$
- Given: sets A_v with $|A_v| = f(v)$
- Goal: find B_v with $|B_v| = g(v)$ and $B_u \cap B_v = \emptyset$ whenever u,v adjacent
- Define $W := \{v \in V \mid g(v) > 0\}$
- Choose color $c \in \bigcup_{v \in W} A_v$
- Define $S := \{v \in V \mid c \in A_v\}$
- Let K be a kernel of S

- Define functions f‘,g‘: V(D) → $\mathbb{N}$

$$g'(v) = \begin{cases} g(v) - 1 & (v \in K) \\ g(v) & (v \notin K) \end{cases}$$

$$f'(v) = |A_v \setminus \{c\}| = \begin{cases} f(v) - 1 & (c \in A_v) \\ f(v) & (c \notin A_v) \end{cases}$$

- i) holds, check ii)
- We have that $\sum_{u \in N[v]} g'(u) < \sum_{u \in N[v]} g(u)$

$$f'(v) \geq \sum_{u \in N[v\in]} g'(u)$$

- Induction hypothesis $\Rightarrow$ G is (f':g')-choosable
- This means: $\exists B_v' \subseteq A_v \setminus \{c\}$ with $|B_v'| = g'(v)$, s.t. $B_v' \cap B_v' = \emptyset$ if u,v adjacent
- Define $B_v := \begin{cases} B_v' \cup \{c\} & (v \in K) \\ B_v' & (v \notin K) \end{cases}$
- It holds: $|B_v| = g(v)$ and $B_u \cap B_v = \emptyset$ if u,v adjacent
- $\Rightarrow$ G is (f:g)-choosable

■

(m:n)-choosable

G is called (m:n)-choosable if it is (f:g)-choosable for the constant functions f(v)=m and g(v)=n

corollary 1

Let D be a digraph in which

i. the maximum outdegree is n-1
ii. every induced subdigraph has a kernel

Then D is (kn:k)-choosable for every k.

In particular D is n-choosable.

Proof of the bipartite LCC

definitions

- An orientation of a graph G is any digraph having G as underying graph
- Let K be a set of vertices in a digraph. Then K absorbs a vertex v if $N[v] \cap K \neq \emptyset$.
- K absorbs a set S if K absorbs every vertex of S

- Example: a kernel of S is an independent subset of S that absorbs S

the bipartite LCC

Let H be a bipartite multigraph, and G:=L(H).

Suppose that G is n-colorable.

Then G is (kn:k)-choosable for every k.

In particular G is n-choosable.

Proof:

- Let $V := V(G) = E(H)$
- Define $A_x := \{v \in V \mid v$ is incident with x, $x \in V(H)\}$. A_x is called row if $x \in X$, column if $x \in Y$
- If v is a vertex of G, then $R(v)$ is the row and $C(v)$ the column containing v
- Take a proper coloring $f: V \rightarrow \{1,\ldots,n\}$ of V
- Define an orientation D of G in which $u \rightarrow v$ if

  $R(u)=R(v)$ and $f(u)>f(v)$ or

  $C(u)=C(v)$ and $f(u)<f(v)$

- Check i) and ii) of Corollary 1
- i): ✓ (since f is one-to-one on N[v])
- ii): induction on |S|
- Given $S \subseteq V$, show the existence of a kernel in S
- Define

  $T := \{v \in S \mid f(v) < f(u) \text{ whenever } v \neq u \in R(v) \cap S\}$
- T absorbs S
- If T is independent, then it's a kernel ✓

- Assume T is not independent, then it has two elements in the same column, say $v_1, v_2 \in T$ with $C(v_1)=C(v_2)=:C$, $f(v_1)<f(v_2)$
- Choose $v_0 \in C \cap S$ s.t. $f(v_0)<f(u)$ whenever $v_0 \neq u \in C \cap S$
- Induction hypothesis $\Rightarrow$ $S \setminus \{v_0\}$ has a kernel K, and K absorbs v_2, this means $N[v_2] \cap K \neq \emptyset$
- $N[v_0] \cap K \supseteq N[v_2] \cap K \Rightarrow$ K absorbs v_0 $\Rightarrow$ K is a kernel of S

- Corollary 1 $\Rightarrow$ D is (kn:k)-choosable $\forall$k
- $\Rightarrow$ G is (kn:k)-choosable for every k

$\blacksquare$