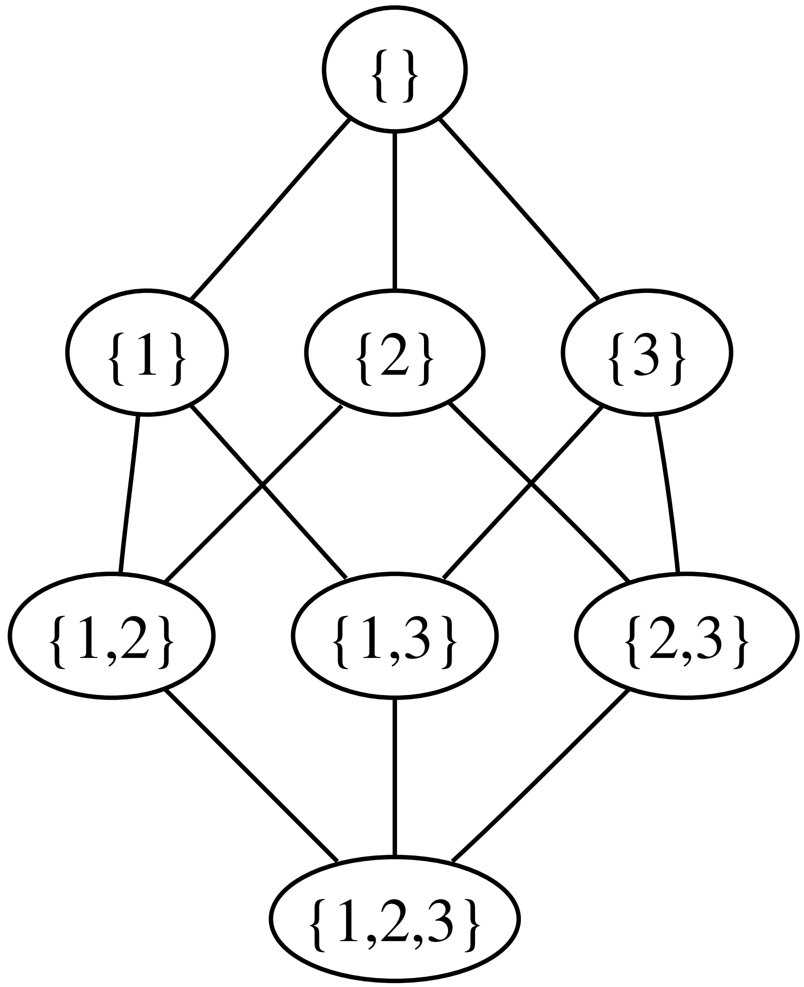
Posets,Chains and Antichains

- A poset (partially ordered set) is a set P together with a binary relation $\leq$ which is transitive ($x < y$ and $y < z$ implies $x < z$) and irreflexive ($x < y$ and $y < x$ cannot both hold)
- x and y are comparable if $x \leq y$ and/or $y \leq x$ hold
- A chain in a poset P is a subset $C \subseteq P$ such that any two elements in C are comparable
- An antichain in a poset P is a subset $A \subseteq P$ such that no two elements in A are comparable

Example

- An important poset is the set 2^X (all subsets of the set X with $|X| = n$) with set inclusion: $x < y$ if $x \subset y$
- This poset can be visualized by a Hasse diagram

Example(cont.)

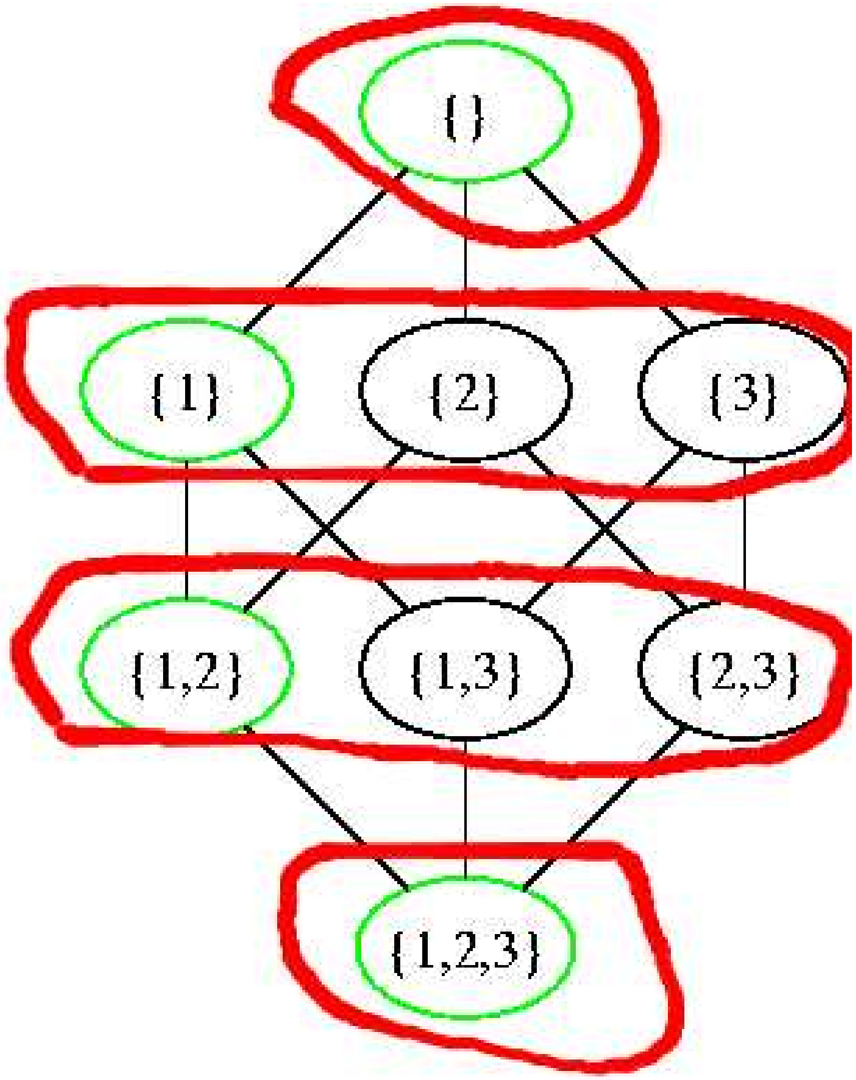


- n=3:

Decomposition of posets using antichains

- Want to partition the poset into antichains
- A poset with a chain of size r cannot be partitioned into fewer than r antichains (Any two elements of the chain must be in a different antichain)
- Theorem: Suppose that the largest chain in the P has size r. Then P can be partioned into r antichains

Example



- n=3

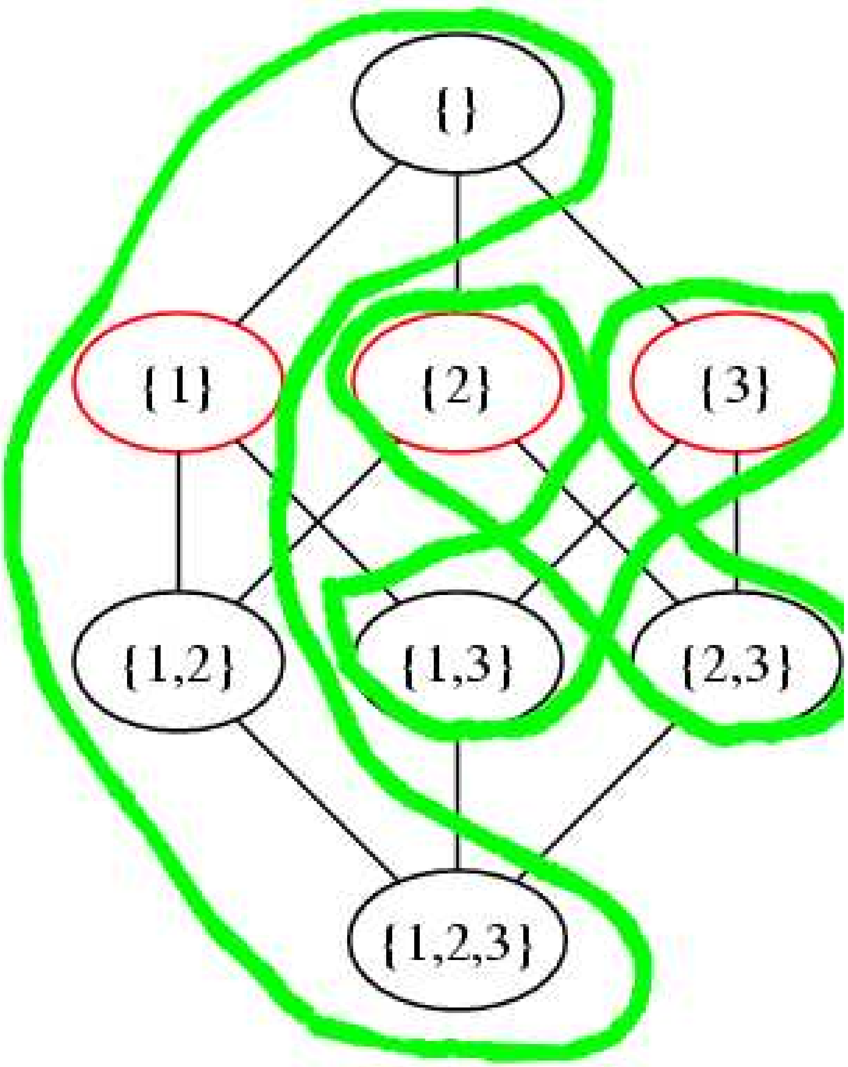
Proof

- Define $l(x)$ as the size of the longest chain whose greatest element is x
- Define A_i as $A_i := \{x : l(x) = i\}$
- $A_1 \cup ... \cup A_r$ is a partition of P into r mutually disjoint sets
- Every A_i is an antichain otherwise there exists two points x, y so that $x < y$ which implies $l(x) < l(y)$

Decomposition of posets using chains

- Want to partition the poset into chains
- A poset with an antichain of size r cannot be partitioned into fewer than r chains (Any two elements of the antichain must be in a different chain)
- Dilworth's theorem: Suppose that the largest antichain in the P has size r. Then P can be partioned into r chains

Example



Proof

- We argue by induction on the cardinality of P
- Let a be a maximal element in P,n=size of the largest antichain in $P' = P - \{a\}$
- P' has according to the induction hypothesis a partition $C_1 \cup ... \cup C_n$
- We show that P has either an antichain of length $n+1$ or it has a partition into n chains
- Let a_i be the maximal element of C_i belonging to an n element antichain

Proof(cont.)

- $A = \{a_1, .., a_n\}$ is an antichain (Transitivity of $\leq$ implies that no two elements from different C_i are comparable)
- If $A \cup \{a\}$ is an antichain we have the partition $C_1 \cup ... \cup C_n \cup \{a\}$ otherwise we have $a > a_i$ for some i

Proof(cont.)

- $K = \{a\} \cup \{x \in C_i : x \leq a_i\}$ is a chain in P and there are no n-element antichains in $P - K$ because of the definition of a_i and so by induction we can decompose $P - K$ into at most $n - 1$ chains

Symmetric chains

- We now consider the poset $2^X := (\{a \subseteq X\}, \leq)$
- The chain $C = \{A_1, ..., A_k\}$ is symmetric if $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1$ for all $i = 1, ..., k-1$
- Theorem: 2^X can be partioned into $\binom{n}{\lfloor n/2 \rfloor}$ mutually disjoint symmetric chains

Proof

- Every chain of the partition contains exactly one set set with $\lfloor n/2 \rfloor$ elements, because the chains are disjoint it follows that there are $\binom{n}{\lfloor n/2 \rfloor}$ chains
- We now show with induction on the cardinality of X that there is a partition at all
- Let x be an arbitrary point in X and $Y = X - \{x\}$

Proof(cont.)

- By induction we can partition 2^Y into symmetric chains $C_1, ..., C_r$
- For every chain $C_i = A_1 \subset ... \subset A_k$ in Y we can produce two chains over X:
  $C_i' = A_1 \subset ... \subset A_{k-1} \subset A_k \subset A_k \cup \{x\}$ and
  $C_i'' = A_1 \cup \{x\} \subset ... \subset A_{k-1} \cup \{x\}$
- C_i' is symmetric since
  $|A_1| + |A_k \cup \{x\}| = |A_1| + |A_k| + 1 = n - 1 + 1 = n$

Proof(cont.)

- C_i'' is symmetric since $|A_1 \cup \{x\}| + |A_{k-1} \cup \{x\}| = |A_1| + |A_{k-1}| + 2 = (n-2) + 2 = n$
- It remains to show that these chains form a partition of X
- If $A \subseteq Y$ then only C_i' contains A
- If $A = B \cup \{x\}$, if B is the maximal element of C_i only C_i' contains A otherwise it is in C_i''

Memory allocation

- Let L be a sequence $L = (a_1, a_2, ..., a_m)$ of not necessarily distinct elements of a set X
- L *contains* a subset A of X if $A = (a_i, a_{i+1}, ..., a_{i+|A|-1})$ for some i
- A sequence L is *universal* if it contains all the subsets of X

An upper bound for the length of an universal sequence

- A trivial upper bound can be obtained by concatenating all subsets
- The resulting sequence has length $n/2 \cdot 2^n$
- Theorem(Lipski):There is a universal sequence for $X = \{1, ..., n\}$ of length at most $\frac{2}{\pi}2^n$

Proof

- We prove the theorem for even n, $n = 2k$
- Let $S = \{1, ...k\}$ be the set of the first k elements and $T = \{k+1, ..., 2k\}$ the set of the last k elements
- The posets corresponding to S and T can be decomposed into $m = \binom{k}{k/2}$ symmetric chains: $2^S = C_1 \cup ... \cup C_m$ and $2^T = D_1 \cup ... \cup D_m$

Proof(cont.)

- We associate the sequence $C_i = (x_1, x_2, ..., x_h)$ to the chain $C_i = \{x_1, ..., x_j\} \subset \{x_1, ..., x_j, x_{j+1}\} \subset ... \subset \{x_1, ..., x_h\}$ where $j + h = k$. A similar association of sequences to chains can be done for the chains D_i
- Every subset A of S is contained in a $C_j = (x_1, x_2, .., x_h)$: $A = \{x_1, x_2, ..., x_{|A|}\}$

Proof(cont.)

- Every subset A of T is contained in a $\bar{D}_j = (x_h, x_{h-1}, .., x_1)$: $A = \{x_1, x_2, ..., x_{|A|}\}$
- Every subset A of X is contained in a sequence $\bar{D}_j C_i$ from some i and j
- Thus if we concatenate all possible $\bar{D}_j C_i$ sequences every subset A is contained in the resulting sequence L

Proof(cont.)

- The length of L is at most km^2
- By Stirling's formula $\binom{k}{k/2} \backsim 2^k\sqrt{\frac{2}{k\pi}}$ the length of L is $km^2 \backsim k\frac{2}{k\pi}2^{2k} = \frac{2}{\pi}2^n$

LYM

- LYM inequality: Let $\mathcal{F}$ be an antichain over a set X of n elements then $\sum_{A \in F} \binom{n}{|A|}^{-1} \leq 1$
- Corollary(Sperner's theorem): Let $\mathcal{F}$ be a family of subsets of an n element set. If $\mathcal{F}$ is an antichain then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$
- Proof:By noting that an expression $\binom{n}{|A|}$ is maximized if $|A| = \lfloor n/2 \rfloor$ we get $|\mathcal{F}| \cdot \binom{n}{\lfloor n/2 \rfloor}^{-1} \leq \sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} \leq 1$

Proof(cont.)

- We associate with every $A \subseteq X$ a permutation on X
- A permutation $(x_1, x_2, ..., x_n)$ of X contains A if $(x_1, ..., x_a) = A$
- Note that there are $a!(n-a)!$ permutations which contain A
- Because in $\mathcal{F}$ no set is a subset of another set every permutation contains at most one $A \in \mathcal{F}$
- Therefore $\sum_{A \in \mathcal{F}} a!(n-a)! \leq n!$

Bollobas's theorem

- Theorem (Bollobas): Let $A_1, ..., A_m$ and $B_1, ..., B_m$ be two sequences of sets such that $A_i \cap B_j = \emptyset$ iff $i = j$ then $\sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} \leq 1$ where $a_i = |A_i|$ and $b_i = |B_i|$
- Theorem: Let $A_1, ..., A_m$ and $B_1, ..., B_m$ be finite sets such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ if $i < j$. Also suppose that $|A_i| \leq a$ and $|B_i| \leq b$ then $m \leq \binom{a+b}{a}$

Proof

- Let X be the union of all sets $A_i \cup B_i$
- A permutation $x_1, ..., x_n$ seperates a pair of disjoint sets (A, B) if $x_k \in A$ and $x_l \in B$ imply $k < l$

Proof(cont.)

- Every permutation seperates at most one of the pairs $(A_i, B_i), i = 1...m$
- Proof: Suppose that a permutation seperates two pairs (A_i, B_i) and (A_j, B_j) with $i \neq j$ and assum w.l.o.g that max$\{k : x_k \in A_i\} \leq max\{k : x_k \in A_j\}$
- The permutation seperates (A_j, B_j) so it follows $min\{l : x_l \in B_j\} > max\{k : x_k \in A_j\} \geq max\{k : x_k \in A_i\}$ which implies $A_i \cap B_j = \emptyset$

Proof(cont.)

- We now estimate the number of permutations seperating one pair (A_i, B_i)
- We have $\binom{n}{a_i+b_i}$ possibilities to select the positions of $A_i \cup B_i$ in the permutation
- There are $a_i!$ possibilities to order A_i and $b_i!$ possibilities to order B_i
- All elements not in $A_i \cup B_i$ can be chosen without restriction so there are $(n - a_i - b_i)!$ possibilities

Proof(cont.)

- So we have $\binom{n}{a_i+b_i} a_i! b_i! (n - a_i - b_i)! = n! \binom{a_i+b_i}{a_i}^{-1}$ permutations which seperate a pair (A_i, B_i)
- Summing up over all m pairs we get the bound: $\sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} \leq n!$

Union-free families

- A family of sets $\mathcal{F}$ is called r-union-free if $A_0 \not\subseteq A_1 \cup A_2 \cup \ldots \cup A_r$ for all distinct $A_0, \ldots, A_r$
- If $\mathcal{F}$ is an antichain then it is 1-union-free
- Theorem(Füredi): Let $\mathcal{F}$ be a family of subsets of an n-element set X and $r \geq 2$. If $\mathcal{F}$ is r-union-free then $|\mathcal{F}| \leq r + \binom{n}{t}$ where $t := \left\lceil (n-r)/\binom{r+1}{2} \right\rceil$

Proof

- Define $\mathcal{F}_t$ as $\{A \mid A \in \mathcal{F}$ and there exists a t-element subset $T \subseteq A$ s.t. T$\not\subseteq A'$ for every other $A' \in \mathcal{F}\}$
- Let $\mathcal{T}_t$ be the family of such t-element subsets
- Let $\mathcal{F}_0$ be $\{A \in F : |A| < t\}$
- Let $\mathcal{T}_0$ be the family of all t-element subsets containing a set in $\mathcal{F}_0$

Proof(cont.)

- $\mathcal{F}$ is r-unionfree for $r \geq 2$ this implies that $\mathcal{F}$ and every subset of it (including $\mathcal{F}_0$) are antichains
- $\mathcal{T}_0$ and $\mathcal{T}_t$ are disjoint
- Proof : Assume a common element B. Then there exists $A, A' \in F$ s.t. $B \subseteq A$ and $A' \subset B$ which implies A and A' are comparable which contradicts the antichain property of $\mathcal{F}$

Proof(cont.)

- We will show that the family $\mathcal{F}' := \mathcal{F} - (\mathcal{F}_0 \cup \mathcal{F}_t)$ has at most r members
- Together with $|\mathcal{F}_0 \cup \mathcal{F}_t| \leq \binom{n}{t}$ (which we will not prove) this proves the theorem
- A is in $\mathcal{F}'$ iff $A \in \mathcal{F}$,$|A| \geq t$ and for every t-subset $T \subseteq A$ there is an $A' \in \mathcal{F}$ s.t. $A' \neq A$ and $A' \supseteq T$

Proof(cont.)

- $A \in \mathcal{F}'$, $A_1, A_2, ..., A_i \in \mathcal{F}$ where $i \leq r$ imply $|A - (A_1 \cup ... \cup A_i)| \geq t(r - i) + 1$
- We proof this statement by assuming the opposite then we can write the set $A - (A_1 \cup ... \cup A_i)$ as the union of $(r - i)$ $t-$element sets $T_{i+1}, ...T_r$
- A can be written as the union of $A_1, ..., A_r$ and $T_{i+1}, ..., T_r$
- By the properties of A every T_j lies in an A_j different from A

Proof(cont.)

- Therefore $A \subseteq A_1 \cup \ldots \cup A_r$ which is a contradiction to the r-union-free property of $\mathcal{F}$
- Suppose $\mathcal{F}'$ has more than r members and take any $r+1$ of them $A_0, A_1, \ldots, A_r \in \mathcal{F}$
- The union of all A_i can be written as $|A_0| + |A_1 - A_0| + |A_2 - (A_0 \cup A_1)| + \ldots + |A_r - (A_0 \cup A_1 \cup \ldots \cup A_{r-1})| \geq (tr+1) + (t(r-1)+1) + (t(r-2)+1) + \ldots + (t0+1) = t\frac{r(r+1)}{2} + r + 1 = t\binom{r+1}{2} + r + 1$

Proof(cont.)

- By the choice of t the right-hand side exceeds the total number of n points which is impossible
- So $\mathcal{F}'$ has at most r elements