.11 Density and Universality

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Book: Stasys Jukna - Extremal Combinatorics

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Definitions

Projections

The **projection** of a vector $v = (v_1,\ldots,v_n)$ onto a set of coordinates $S = \{i_1,\ldots,i_k\}$ is the vector $\mathbf{v|_S} := (v_{i_1},\ldots,v_{i_k})$.

The projection of a set of vectors $A \subset \mathbf{\{0,1\}^n}$ is the set $\mathbf{A|_S} := \{v|_S \; v \in A\}$.

Dense and Universal Sets

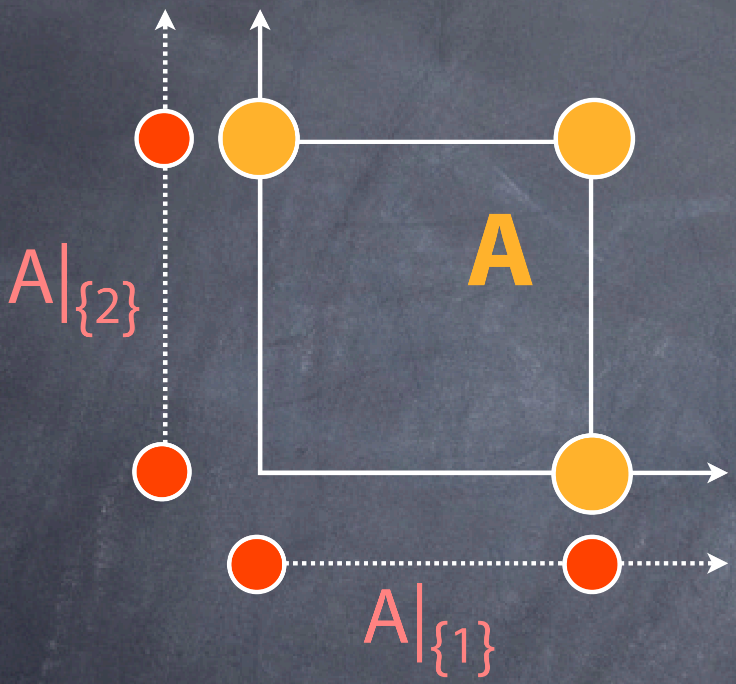
A vector set $A \subseteq \{0,1\}^n$ is called **(n,k)-universal** if the projection of A onto any subset S of k coordinates contains all possible 2^k configurations.

$$\forall S \quad |S| = k \quad A|_S = \{0,1\}^k$$

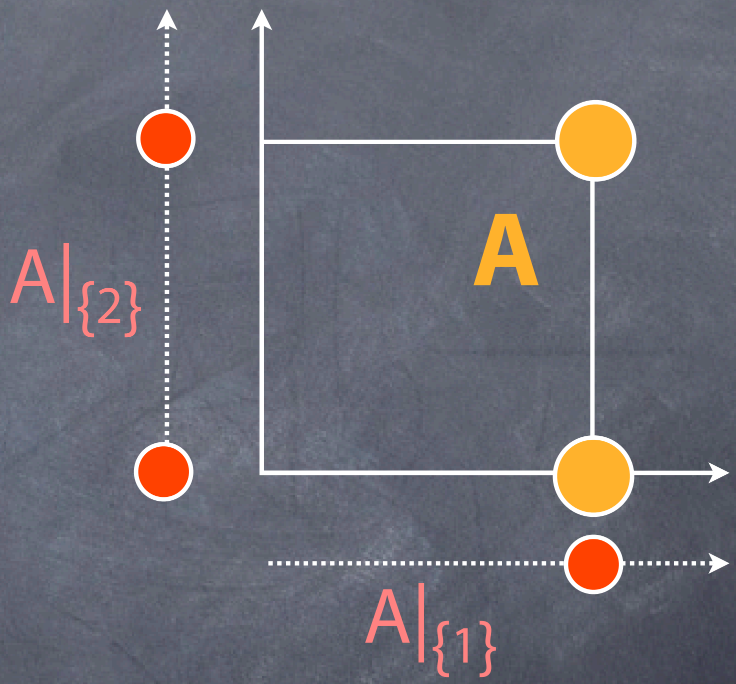
A is called **(n,k)-dense** if the same holds not for all but for at least one subset S of k indices.

$$\exists S \quad |S| = k \quad A|_S = \{0,1\}^k$$

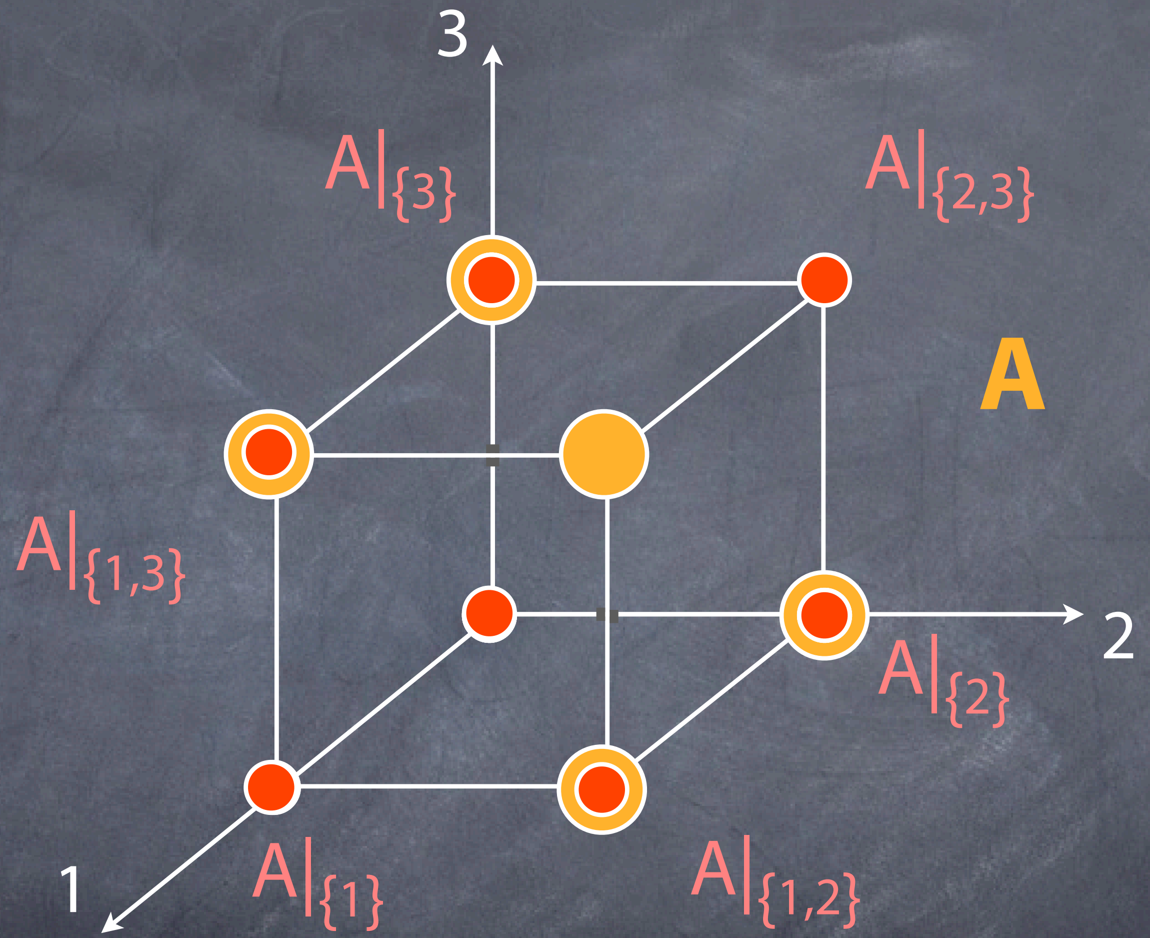
Examples



A is (2,1)-dense
and (2,1)-universal



A is (2,1)-dense
but not (2,1)-universal



A is (3,1)-dense
and (3,1)-universal

A is (3,2)-dense
but *not* (3,2)-universal

Dense sets

What can one say about the size of dense sets ?

- $|A| \geq 2 \;\Rightarrow\;$ A (n,1)-dense
- A (n,k)-dense $\;\Rightarrow\; |A| \geq 2^k$
- There are $\sum_{i=0}^{k-1} \binom{n}{i}$ vectors in $\{0,1\}^n$ which have less than k ones. These are obviously not (n,k)-dense.

$$\boxed{H(n,k) := \sum_{i=0}^{k-1} \binom{n}{i}}$$

Theorem 1: (Perles and Shelah 1972 / Sauer 1972 / Vapnik and Chervonenkis 1971)

$$A \subseteq \{0,1\}^n \text{ and } |A| > H(n,k) \quad \Rightarrow \quad A \text{ (n,k)-dense}$$

Proof Induction on n:

Base cases:

$|A| > H(n,1) = 1$ means A contains at least two different vectors and hence is (n,1)-dense.

$|A| > H(n,n) = 2^n - 1$ means A is the entire n-cube and hence is (n,n)-dense.

$n-1 \to n$:

Let **B** be the projection of A onto the first n-1 coordinates, and **C** the set of all vectors u in $\{0,1\}^{n-1}$ for which both vectors (u,0) and (u,1) belong to A.

$$|A| = |B| + |C|$$

- If $|B| > H(n-1,k)$ then B is (n-1,k)-dense by induction hypothesis, and hence the whole set A is also (n,k)-dense.

- If $|B| \le H(n-1,k)$ then

$$|C| = |A| - |B| > H(n,k) - H(n-1,k)$$

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

$$= \sum_{i=0}^{k-1}\binom{n}{i} - \sum_{i=0}^{k-1}\binom{n-1}{i}$$

$$= \sum_{i=1}^{k-1}\binom{n-1}{i-1}$$

$$= \sum_{j=0}^{k-2}\binom{n-1}{j} = H(n-1,k-1)$$

So C is (n-1,k-1)-dense by induction hypothesis. Because $C \times \{0,1\}$ lies in A, the whole set A is also (n,k)-dense. ∎

Universal sets

Isolated neighbor condition

Property of (bipartite) graphs which is equivalent to the universality property of 0-1 vectors.

A vector set $A \subset \{0,1\}^n$ is called **(n,k)-universal** if the projection of A onto any subset of k coordinates S contains all possible 2^k configurations.

$$A|_S = \{0,1\}^k \quad \forall\, S \quad |S| = k$$

Bipartite graphs

Given a bipartite graph with parts of size n: $G=(V_1,V_2,E)$ with $|V_1|=|V_2|=n$

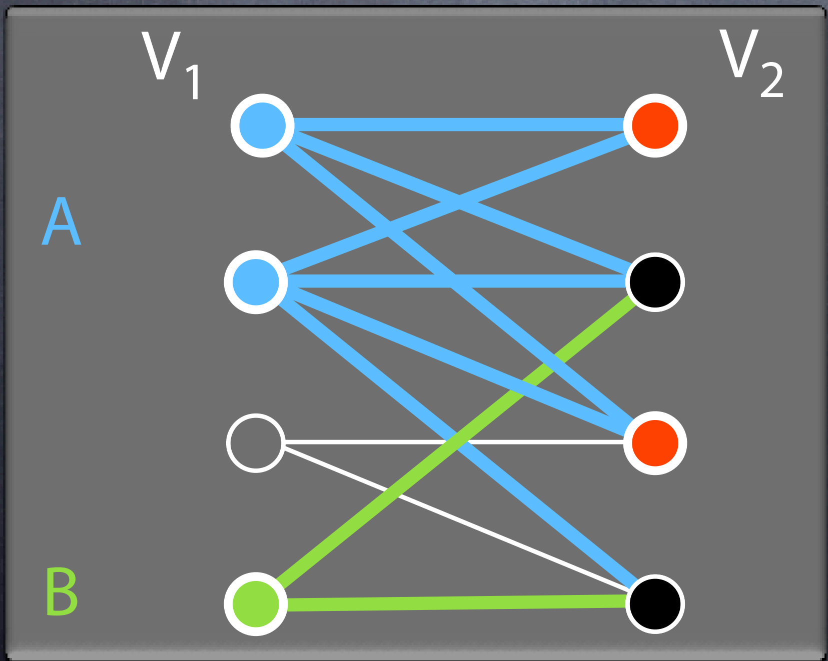
we say:

A node $x \in V_2$ is a **common neighbor** for a set of nodes $A \subseteq V_1$ if x is jointed to each node of A.

A node $x \in V_2$ is a **common non-neighbor** for a set of nodes $B \subseteq V_1$ if x is jointed to no node of B.

$A, B \subseteq V_1$

v(A,B) := # nodes in V_2 that are common neighbors of A and common non-neighbors of B at the same time.



v(A,B) = 2

(Gál 1998)

A bipartite graph $G = (V_1, V_2, E)$ satisfies the **isolated neighbor condition for k** if

$$\mathbf{v(A,B) > 0}$$

for any two disjoint subsets $A, B \subseteq V_1$ such that $|A| + |B| = k$.

Proposition 2:

Let G be a bipartite Graph with parts of size n and C be the set of columns of its adjacency matrix. then:

G satisfies the isolated neighbor condition for k

$\Leftrightarrow$ C is (n,k)-universal.

Proof of Proposition 2:

Let $M = (m_{x,y})$ be the adjacency matrix of G.

$$m_{x,y} = \begin{cases} 1 & (x,y) \in E \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$

nodes of V_1

Let $S = \{i_1,\ldots,i_k\}$ be a subset of k rows of M.

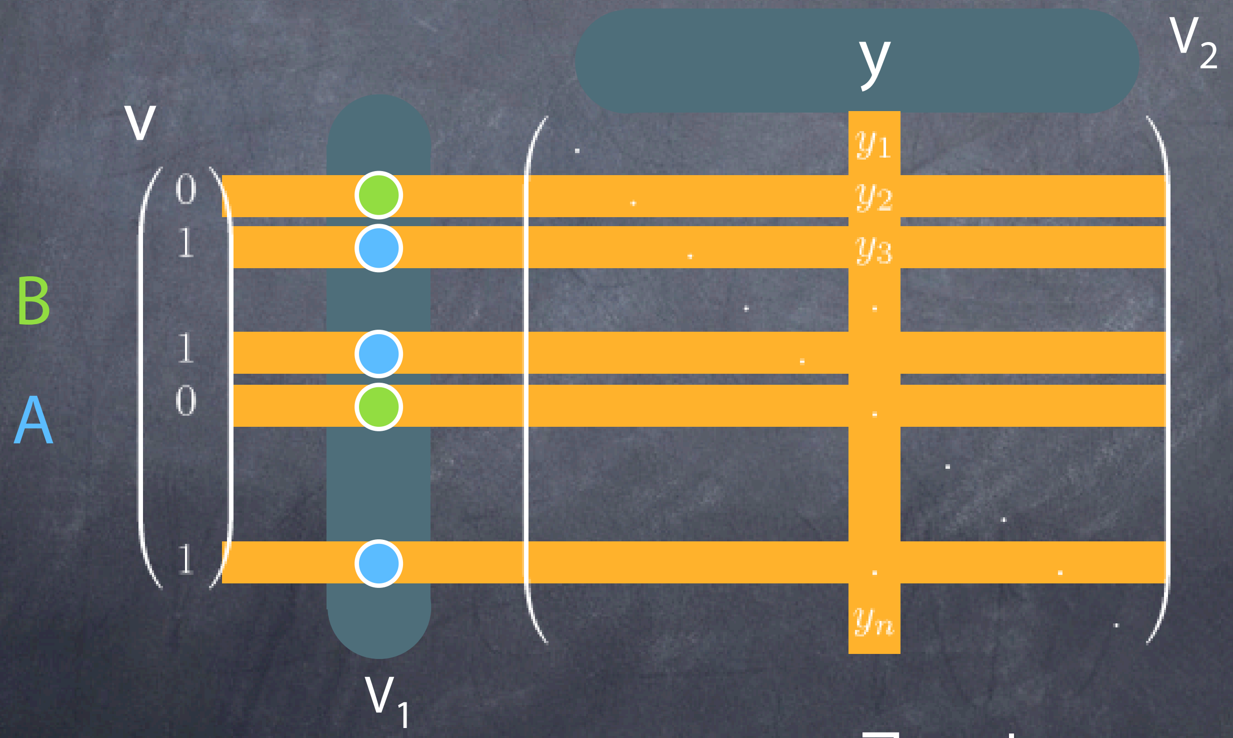
Let $v = (v_{i_1},\ldots,v_{i_k})$ be an arbitrary vector in $\{0,1\}^k$.

Is there a column y of M with $y|_S = v$?

Split our S into two subsets A and B

$j \in A$ iff $v_j = 1$, $j \in B$ iff $v_j = 0$

$|A| + |B| = |S| = k$



$\exists \quad y|_S = v$

Construction of small universal sets

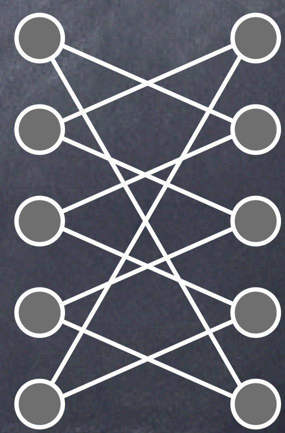
Paley graphs

A bipartite **Paley graph** is a bipartite graph $G_p = (V_1, V_2, E)$ with parts $V_1 = V_2 = \mathbb{F}_p$.

$(x,y) \in E$ iff $x-y$ is a non-zero square in $\mathbb{F}_p$.

Where p is a prime number.

G_5



Theorem 3: (Bollobás and Thomason 1981)

Let $\mathbf{G_p} = (V_1,V_2,E)$ be a bipartite Paley graph with $p \geq 9$, and A, B disjoint sets of nodes in V_1 such that $|A| + |B| = k$. then

$$\left|v(A,B) - 2^{-k}p\right| \leq k\sqrt{p}$$

in particular, $v(A,B) > 0$ as long as $k2^k < \sqrt{p}$

$$\Leftrightarrow \quad k\sqrt{p} < 2^{-k}p$$

Quadratic residue character:

- $\chi(x) := \begin{cases} 1 & \exists\, a \in F_p^* \quad x = a^2 \\ 0 & x = 0 \\ -1 & \nexists\, a \in F_p \quad x = a^2 \end{cases} \qquad x \in \mathrm{F}_p$
- $\chi(x) = x^{(p-1)/2} \qquad \forall\, x \in \mathrm{F}_p$
- $\chi(x \cdot y) = \chi(x) \cdot \chi(y) \qquad \forall\, x,y \in \mathrm{F}_p$

Theorem (Weil 1948):

Let f(x) be a polynomial over F_p which is not the square of another polynomial and has precisely s distinct zeros. then:

$$\left| \sum_{x \in \mathbb{F}_p} \chi(f(x)) \right| \leq (s-1)\sqrt{p}$$

Proof of Theorem 3:

Remember that (x,y) is an edge in G_p $\Leftrightarrow$ $\chi_{(x-y)} = 1$.

We say $x' \in V_2$ is the **copy** of $x \in V_1$ if both these nodes correspond to the same element in $\mathbb{F}_p$.

No x is joined to its copy x′.

We define

U $:= V_2 - (A' \cup B')$ where $A', B' \subseteq V_2$ are the copies of $A, B \subseteq V_1$

g(x) $:= \prod_{a\in A}(1+\chi(x-a))\prod_{b\in B}(1-\chi(x-b))$ for nodes $x \in V_2$

For each node $x \in U$, g(x) is non-zero iff x is joined to every node in A and to no node in B, in which case it is precisely 2^k. So

$$\sum_{x\in U} g(x) = 2^k\, v^*(A,B)$$

where $v^*(A,B)$ is the number of nodes in U which are joined to every node of A and to no node of B.

$$g(x) = \prod_{c\in A\cup B}(1 \pm \chi(x-c))$$

$+$: $c \in A$

$-$: $c \in B$

$$= \quad 1 + \sum_{C\subseteq A\cup B}(-1)^{|C\cap B|}\chi(f_C(x))$$

where $f_C(x) = \prod_{c\in C}(x-c)$

$$\left|\sum_{x\in\mathbb{F}_p} g(x) - p\right| = \left|\sum_{x\in\mathbb{F}_p} (g(x)-1)\right| = \left|\sum_{x\in\mathbb{F}_p} \sum_{C} (-1)^{|C\cap B|}\,\chi(f_C(x))\right|$$

$$\leq \left|\sum_{C} (-1)^{|C\cap B|} \sum_{x\in\mathbb{F}_p} \chi(f_C(x))\right| \leq \sum_{C}\left|\sum_{x\in\mathbb{F}_p} \chi(f_C(x))\right|$$

$$\leq \sum_{C} (|C|-1)\sqrt{p}$$

(Weil's theorem)

$$= \sqrt{p}\sum_{s=1}^{k}\binom{k}{s}(s-1) = \sqrt{p}\left(\sum_{s=1}^{k}\binom{k}{s}s - \binom{k}{s}\right)$$

$$\boxed{\sum_{s=1}^{k} s\binom{k}{s} = k\,2^{k-1}}$$

$$= \sqrt{p}\,\left(k\,2^{k-1} - (2^k - 1)\right)$$

Because for every $x \in A' \cup B'$, $g(x) \leq 2^{k-1}$, or

$$\left|\sum_{x\in A'\cup B'} g(x)\right| \leq k\,2^{k-1}$$

we have

$$\left|\sum_{x\in U} g(x) - p\right| \leq \sqrt{p}\,(k\,2^{k-1} - 2^k + 1) + k\,2^{k-1}$$

By dividing both sides by 2^k and using $\left(\sum_{x \in U} g(x) = 2^k \, v^*(A,B) \right.$ we get:

$$\left| v^*(A,B) - 2^{-k} p \right| \leq \sqrt{p} \left(\frac{k}{2} - 1 + \frac{1}{2^k} \right) + \frac{k}{2}$$

and because $\left| v(A,B) - v^*(A,B) \right| \leq |A' \cup B'| = k$:

$$\left| v(A,B) - 2^{-k} p \right| \leq \frac{k\sqrt{p}}{2} - \cancel{\sqrt{p} + \frac{\sqrt{p}}{2^k}} + \frac{k}{2} + k$$

for $p > 9$

$$\leq k\sqrt{p}$$

$\blacksquare$

Theorem 3 together with Proposition 2 gives us,

for every prime $n > 9$

and for every k such that $k\,2^k < \sqrt{n}$,

an explicit construction of (n,k)-universal sets of size **n**.

Using linear codes it is possible to construct such sets of size $n^{O(k)}$ for arbitrary k.

Full graphs

A graph containing of order n is called **k-full** iff it contains every k-vertex graph as a subgraph.

How many vertices a graph must have to be k-full ?

If G is a k-full graph of order then $\binom{n}{k}$ is at least the number of non-isomorphic subgraphs of order k.

$$\binom{n}{k} \geq \frac{2^{\binom{k}{2}}}{k!} \quad \Leftrightarrow n(n-1)...(n-k+1) \geq 2^{\frac{k(k-1)}{2}}$$

$$\Rightarrow n \geq 2^{\frac{(k-1)}{2}}$$

Construction of small k-full graphs

Let $\mathbf{P_k}$ be the graph of order $n = 2^k$ whose vertices are the subsets of $\{1,\dots,k\}$, and where two vertices are joined iff $|A \cap B|$ is even.
Exception: A is joined to $\{\}$ iff $|A|$ is even.

Theorem 4: (Bollobás and Thomason 1981)

The graph $\mathbf{P_k}$ is k-full.

Proof of Theorem 4:

Let G be an arbitrary graph with vertex set $\{v_1,v_2,\ldots,v_k\}$. We claim there are sets $A_1,A_2,\ldots,A_k$ uniquely determined by G such that

$|A_i \cap A_j|$ is even iff v_i and v_j are joined in G

$A_i \subseteq \{1,\ldots,i\}$

Set $A_1 := \{\}$

Suppose we already have choosen $A_1,A_2,\ldots,A_{i-1}$.

We search for an A_i which is properly joined to all the sets $A_1,A_2,\ldots,A_{i-1}$, that is, $|A_i \cap A_j|$ must be even iff v_i is joined to v_j in G.

We will obtain A_i as the last set in the sequence $B^i_1 \subseteq B^i_2 \subseteq \dots \subseteq B^i_{i-1} = A_i$ with,
for each $1 \leq j < i$, B^i_j is a set properly joined to all sets $A_1, A_2, \dots, A_j$.

$$B^i_0 := \{i\}$$

For $j \geq 1$:

If v_j is joined to v_i set $B^i_j := B^i_{j-1}$ or $B^i_j := B^i_{j-1} \cup \{j\}$ depending on $|B^i_{j-1} \cap A_j|$.

If v_j is not joined to v_i we act dually.

Our choice of whether j is in B^i_j does effect $|B^i_j \cap A_j|$ (since $j \in A_j$),
but none of $|B^i_j \cap A_k|$ $k<j$ (since $A_k \subseteq \{1,\dots,k\}$)

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