Intersecting Families

Extremal Combinatorics
Philipp Zumstein

A familiy of sets is **intersecting** if any two of its sets have a non-empty intersection.

Let $\mathcal{F}$ be an intersecting family of subsets of {1,...,n} = [n].

Question: How large can such a family be?

Take all subsets containing a fixed element.

This is an intersecting family with

$$|\mathcal{F}| = 2^{n-1}$$

Can we find larger intersecting families? **No!**

A set an its complement cannot both be members of $\mathcal{F}$

So we get:

$$|\mathcal{F}| \leq 2^{n-1}$$

A familiy of sets is **intersecting** if any two of its sets have a non-empty intersection.

Let $\mathcal{F}$ be an intersecting family of k-element subsets of {1,...,n} = [n].

Question: How large can such a family be?

Trivial upper bound:

$$|\mathcal{F}| \leq \binom{n}{k}$$

First case: n < 2k:

- Every pair of k-element subsets of [n] has a non-empty intersection.
- So we could choose $\mathcal{F}$ as the set of all k-element subsets of [n]
- So the trivial upper bound is sharp.

Second case: n ≥ 2k

Take all the k-element subsets containing a fixed element.

Examples: n = 5, k = 2, fix the element 1

{1,2}, {1,3}, {1,4}, {1,5}

n = 5, k = 3, fix the element 1

{1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,3,5}, {1,4,5}

This is an intersecting family with

$$|\mathcal{F}| = \binom{n-1}{k-1}$$

Can we find larger intersecting families?

Theorem: (Erdős-Ko-Rado, 1961)

If 2k ≤ n then every intersecting family of k-element subsets of an n-element set has at most $\binom{n-1}{k-1}$ members.

Proof: (due to G.O.H. Katona, 1972)

W.l.o.g. we can assume X = {0,1,...,n-1}.

For s∈X, define

$$B_s := \{s, s+1, \dots, s+k-1\} \subseteq X,$$

where the addition is modulo n.

Claim: At most k of the sets B_s can belong to $\mathcal{F}$

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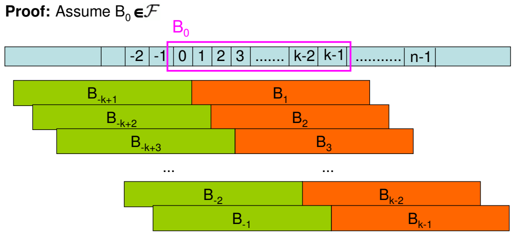
Proof: Assume $B_0 \in \mathcal{F}$



There are 2k-2 sets that intersect with B_0. These sets can partioned into k-1 pairs of disjoint sets B_i, B_{i+k}, where $-(k-1) \le i \le -1$.

Since $\mathcal{F}$ can contain at most one set of each pair the assertion of the claim follows. □

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$$B_s := \{s, s+1, \dots, s+k-1\} \subseteq X,$$

where the addition is modulo n.

Claim: At most k of the sets B_s can belong to $\mathcal{F}$

L := „number of pairs (π,s), where π is a permutation of X and s is a point of X, such that the set $\pi(B_s) = \{\pi(s), \pi(s+1), \dots, \pi(s+k-1)\}$ belongs to $\mathcal{F}$"

Double counting:

$$L \le kn!$$
$$L = n|\mathcal{F}|k!(n-k)!$$

Together:

$$|\mathcal{F}| \le \binom{n-1}{k-1}$$ ■

Summary

A familiy of sets is **intersecting** if any two of its sets have a non-empty intersection.

Let $\mathcal{F}$ be a family of subset of {1,...,n} = [n].	Let $\mathcal{F}$ be a family of k-element subsets of {1,...,n} = [n].	
Question: How large can such a family be? (Maximum)		
	n < 2k:	n ≥ 2k:
$|\mathcal{F}| \le 2^{n-1}$	$|\mathcal{F}| \le \binom{n}{k}$	$|\mathcal{F}| \le \binom{n-1}{k-1}$ Erdős-Ko-Rado

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Projective planes

A **projective plane of order q** consists of a set X of elements called points and a family L of subsets of X called lines having the following properties:

1. Each pair of points determines a unique line.
2. Each two lines intersect in exactly one point.
3. Any point lies on q+1 lines.
4. Every line has q+1 points.
5. There are q^2+q+1 points.
6. There are q^2+q+1 lines.

A **projective plane of order q** consists of a set X of elements called points and a family $\mathcal{L}$ of subsets of X called lines having the following properties:
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(ii) Every line has q+1 points.
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Proposition:
A projective plane of order q has the following properties:
(a) Any point lies on q+1 lines.
(b) There are q^2+q+1 lines.
(c) Each two lines intersect in exactly one point.

Proof: (a) Take a point x
There are q(q+1) other points (iii)
Each line through x contain q further points (ii)
Two such lines don't overlap (apart from x) (i)
Each point lies on a line through x (i)
So, there are exactly q+1 lines through x.

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Proof: (b) Counting the pairs (x,L) with $x \in L$ in two ways:

$$\begin{aligned}\sharp\{(x,L); x \in L\} &= |X|(q+1) = (q^2+q+1)(q+1)\\ &= |\mathcal{L}|(q+1)\end{aligned}$$

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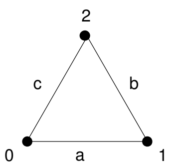
Proof: (c) Let L_1 and L_2 be lines, and x a point of L_1 (and not L_2).
Then the q+1 points from L_2 are joined to x by different lines.
x lies on exactly q+1 lines.
So one of this lines has to be L_1.
But then, L_1 and L_2 intersect in exactly one point.

■

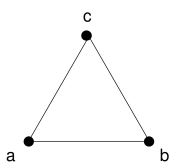
Example and Duality

q = 1:

Points: X = { 0,1,2 }
Lines: $\mathcal{L}$ = { {0,1}, {1,2}, {2,0} }



projective plane of order 1



dual projective plane

The construction

Let $q = p^r$ where p is prim and r is an positive integer.
Look at field $\mathbb{F}_q$ = GF(q) =: K
And the vectorspace K^3

We define our **points** as 1-dimensional subspaces of K^3, i.e.

$$[x_0, x_1, x_2] := \{(cx_0, cx_1, cx_2); c \in \mathbb{F}_q, c \neq 0\}$$

for $(x_0,x_1,x_2) \in V := K^3 - (0,0,0)$.
(Note: If $x_0 = x_1 = x_2 = 0$ then this is a 0-dimensional subspace. So we don't allow this case.)

Such a point is a set of q-1 vectors from V.
There are $(q^3-1) / (q-1) = q^2+q+1$ such points.

This shows condition (iii).

Let $q = p^r$ where p is prim and r is an positive integer.
Look at field $\mathbb{F}_q$ = GF(q) =: K

For $(x_0,x_1,x_2) \in V := K^3 - (0,0,0)$ we define the **points**

$$[x_0, x_1, x_2] := \{(cx_0, cx_1, cx_2); c \in \mathbb{F}_q, c \neq 0\}$$

The **line $L(a_0,a_1,a_2)$**, where $(a_0,a_1,a_2) \in V$, is defined to be the set of all those points $[x_0,x_1,x_2]$ for which

$$a_0x_0 + a_1x_1 + a_2x_2 = 0.$$

Two triples (x_0,x_1,x_2) and (cx_0,cx_1,cx_2) either both satisfy this equation or none does.

How many points does such a line have?
Because $(a_0,a_1,a_2) \in V$, this vector has at least one nonzero component; say $a_0 \neq 0$.
Chose x_1 and x_2 arbitrary not both 0 and (because K is a field) we can uniquely determine x_0.
So we get q^2-1 solutions $(x_0,x_1,x_2) \in V$. These are q+1 points.
This shows (ii).

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Let $[x_0,x_1,x_2]$ and $[y_0,y_1,y_2]$ be two distinct points. How many lines contain both these points? For each such line $L(a_0,a_1,a_2)$

$$a_0x_0 + a_1x_1 + a_2x_2 = 0$$
$$a_0y_0 + a_1y_1 + a_2y_2 = 0$$

Since the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix}$$

has rank 2 (the rows are linearly independent), the solution-space is 1-dimensional, i.e. one point. This shows (i).

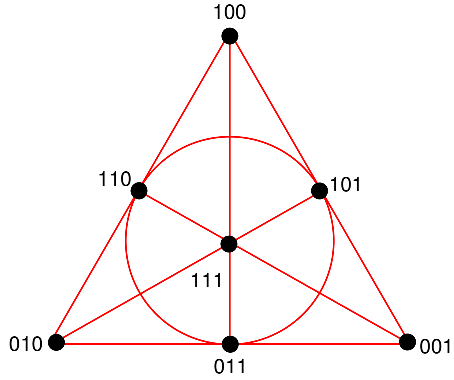
Example: Fano Plane

q = 2
Projective plane with 7 points and 3 points on a line.

K = GF(2),
V = K^3 – 000 = { 001, 010, 011, 100, 101, 110, 111 }
These are also the points.

Lines:

v ∈ V	equation:	line:
001	$x_2 = 0$	L(001) = { 010, 100, 110 }
010	$x_1 = 0$	L(010) = { 001, 100, 101 }
011	$x_1+x_2 = 0$	L(011) = { 011, 100, 111 }
100	$x_0 = 0$	L(100) = { 010, 001, 011 }
101	$x_0+x_2 = 0$	L(101) = { 010, 101, 111 }
110	$x_0+x_1 = 0$	L(110) = { 001, 110, 111 }
111	$x_0+x_1+x_2 = 0$	L(111) = { 011, 101, 110 }



Bruck-Chowla-Ryser Theorem:

If a projective plane of order n exists, where n is congruent 1 or 2 modulo 4, then n is the sum of two squares of integers.

There is no projective plane of order 6 or 14.

What about 10? Is there a projective plane of order 10?
1988: There is no projective plane of order 10

Open Question: Is there a projective plane of order 12?

Summary

A **projective plane** of order q consists of a set X of elements called points and a family L of subsets of X called lines having the following properties:

1. Each pair of points determines a unique line.
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3. Any point lies on q+1 lines.
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5. There are q^2+q+1 points.
6. There are q^2+q+1 lines.

If $q = p^r$ is a power of a prime number, then there exist a projective plane of order q.

Maximal intersecting families

Let $\mathcal{F}$ be a k-uniform family of sets of some n-element set.

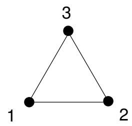
$\mathcal{F}$ is **maximal intersecting** if
(i) is intersecting;
(ii) the addition of any new k-element set to $\mathcal{F}$ destroys this property.

Examples: n = 8, k = 2

$\mathcal{F}$ = { {1,2}, {1,3}, {1,4}, {1,5}, {1,6}, {1,7}, {1,8} }

Can we get a maximal intersecting family with fewer subsets? **Yes!**

$\mathcal{F}$ = { {1,2}, {2,3}, {3,1} }



Maximal intersecting families

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Example: n = 7, k = 3

$\mathcal{F}$ = { {1,2,3}, {1,4,5}, {1,6,7}, {2,4,6}, {2,5,7}, {3,4,7}, {3,5,6} }

{4,5,6}, {2,3,6}, {2,3,4}, {1,3,5}, {1,3,4}, {1,2,5}, {1,2,4}
{4,5,7}, {2,3,7}, {2,3,5}, {1,3,7}, {1,3,6}, {1,2,6}, {1,2,7}
{4,6,7}, {2,6,7}, {2,4,5}, {1,5,7}, {1,4,6}, {1,5,6}, {1,4,7}
{5,6,7}, {3,6,7}, {3,4,5}, {3,5,7}, {3,4,6}, {2,5,6}, {2,4,7}

Is this family maximal intersecting? **Yes!**

Example: n = 7, k = 3

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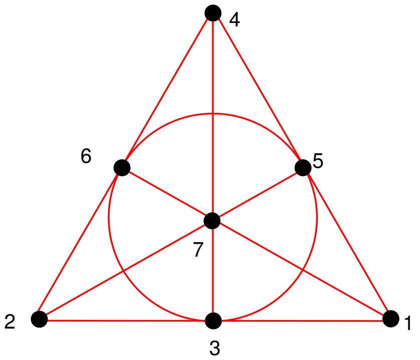


<u>One case: n < 2k:</u>

The only maximal intersecting family is the family of all k-element subsets.

<u>Another case: n ≥ k^2-k+1:</u>

Consider the family $\mathcal{F}$ of lines in a projective plane of order k-1.

There are k^2-k+1 lines and each line is a k-element subset of an n-element set of points. Any two lines intersect in precisely one point; so $\mathcal{F}$ is intersecting.

Claim: $\mathcal{F}$ is maximal intersecting

Proof: (indirect) Let E be a k-element set which intersects all the lines.
Assume E is not a line (i.e. E is not a member of $\mathcal{F}$).
Take two points x ≠ y of E.
Let L be the line through x and y.
Take a point z ∈ L - E.
z belongs to k lines, and each of them intersect E.
The intersection with L contains at least 2 elements x,y.
|E| > k. ■

Theorem (Füredi, 1980):

Let $\mathcal{F}$ be a maximal intersecting family of k-element sets of an n-element set. Then

(i) $|\mathcal{F}| \geq \frac{1}{2}\left(\frac{n}{n-k}\right)^k$

(ii) In particular: $|\mathcal{F}| > k^2$ for $n \leq \frac{k^2}{2\log k}$

Proof: (i) N = „number of pairs (F,E) where F∈$\mathcal{F}$ and E is a k-element subset disjoint from F (and hence, E ∉ $\mathcal{F}$)"

Double counting!

$$N \geq \binom{n}{k} - |\mathcal{F}|$$

$$N \leq |\mathcal{F}| \cdot \binom{n-k}{k}$$

Together

$$|\mathcal{F}| \geq \frac{1}{2}\left(\frac{n}{n-k}\right)^k$$

(ii) The (stronger) assumption $n \leq \frac{k^2}{1+2\log k}$

leads to $|\mathcal{F}| > k^2$ ■

A Helly-type result

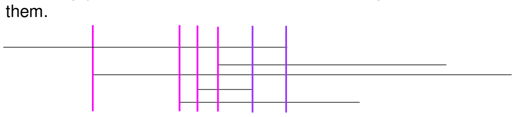
E. Helly, 1923:

If n ≥ k+1 convex sets in $\mathbb{R}^k$ have the property that any k+1 of them have a nonempty intersection, then there is a point common to all of them.

<u>Special case:</u> k = 1:

convex sets in $\mathbb{R}$ = intervals

We take n ≥ 2 such intervals with the property that any two of them have a nonempty intersection. We claim that there is a point common to all of them.



A Helly-type result

Let $\mathcal{F}$ be a family and k be the minimum size of its member. If any k+1 members of $\mathcal{F}$ intersect (i.e. , share a common point) then all of them do.

Proof: Assume the opposite, that the intersection of all sets in $\mathcal{F}$ is empty.

Take a set A = {x_1,...,x_k} ∈ $\mathcal{F}$ of minimum size.

x_i is not in every set of $\mathcal{F}$

So, there is a set B ∈ $\mathcal{F}$ such that x_i ∉ B_i

} for every i = 1, ... , k

We get

$$A \cap B_1 \cap \ldots \cap B_k = \emptyset \quad ↯$$

■