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# *Approximation Algorithms and Semidefinite Programming* FS12
# *Exercise Set 2*

**Course Webpage:** `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
**Due date:** March 20, 2012.

## Exercise 1 (Fekete's Lemma)

*[Exercise 3.1]* Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of real number such that for all natural numbers $k$ and $l$,

$$x_{k+l} \geq x_k + x_l\,.$$

Such a sequence is called *super-additive*. Prove that

$$\lim_{k\to\infty} \frac{x_k}{k} = \sup\left\{\frac{x_k}{k} \,:\, k \in \mathbb{N}\right\}$$

where both the limit and the supremum may be unbounded.

## Exercise 2 (A trace inequality)

Let $X, Y \in \mathrm{PSD}_n$ be two positive semidefinite matrices. Show that

$$0 \leq \mathrm{Tr}\big[XY\big]^2 \leq \mathrm{Tr}\big[X^2\big]\mathrm{Tr}\big[Y^2\big]\,.$$

## Exercise 3 (Semidefinite Program in equational form)

*[Exercise 3.2]* Show that the program (3.6) from the book can be rewritten into a semidefinite program in equational form (Definition 2.4.1) with the same value.

$$\begin{array}{ll} \max & t \\ \text{s.t.} & \mathbf{u}_i^T\mathbf{u}_j = 0\,,\ \text{for all } \{i,j\} \in \bar{E} \\ & \mathbf{c}^T\mathbf{u}_i \geq t, i \in V \\ & \|\mathbf{u}_i\| = 1, i \in V \\ & \|\mathbf{c}\| = 1\,. \end{array} \tag{3.6}$$

## Exercise 4 (Shannon capacity of disjoint union)

*[Exercise 3.5]* For two graphs $G, H$, let $G+H$ stand for the disjoint union of $G$ and $H$. Formally, we let $H'$ be an isomorphic copy of $H$ whose vertex set is disjoint from $V(G)$ and we put $V(G+H) = V(G) \cup V(H')$, $E(G+H) = E(G) \cup E(H')$. Prove that $\Theta(G+H) \geq \Theta(G) + \Theta(H)$.