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March 16, 2012

Approximation Algorithms and Semidefinite Programming FS12
Exercise Set 3

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Due date: March 27, 2012.

Exercise 1 (SDP formulation of the Lovász theta function)

Let $G = (V, E)$ be a graph. Show that $\vartheta(G)$ can be expressed as the value of the following optimization problem:

$$\begin{aligned} \vartheta(G) = \min \quad & \lambda_{\max}(\mathbf{1}\mathbf{1}^T + X) \\ \text{s.t.} \quad & X_{ij} = 0\,, \text{ if } \{i,j\} \in \bar{E}, \text{ or } i = j \\ & X \in \mathrm{SYM}_n\,. \end{aligned} \tag{1}$$

Exercise 2 (Dual of direct sum)

[Exercise 4.3] Let V and W be two finite dimensional vector spaces, each equipped with a scalar product and let $K \subset V, L \subset W$ be closed convex cones. Show that

$$(K \oplus L)^* = K^* \oplus L^*\,.$$

Exercise 3 (Dual of the norm cone)

Let $\langle \cdot, \cdot \rangle_{\mathbb{R}^{n-1}}$ denote a scalar product on $\mathbb{R}^{n-1}$. For a norm $\|\cdot\|$ on $\mathbb{R}^{n-1}$ we define its *dual norm* as $\|\mathbf{x}\|_* := \sup_{\|\mathbf{y}\|=1} \langle \mathbf{x}, \mathbf{y} \rangle$. The *norm cone* is defined as

$$K := \{(\mathbf{x}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \|\mathbf{x}\| \le t\}\,.$$

$\mathbb{R}^{n-1} \times \mathbb{R}$ can naturally be endowed with a scalar product:

$$\langle (\mathbf{x}, t), (\mathbf{y}, s) \rangle_{\mathbb{R}^{n-1} \times \mathbb{R}} := \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^{n-1}} + st\,,$$

for $s, t \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n-1}$.

a) Show that the dual cone K is defined by the dual norm, e.g.

$$K^* = \{(\mathbf{x}, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \|\mathbf{x}\|_* \le t\}\,.$$

b) *[Exercise 4.2]* What is the dual of the ice cream cone?