Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Dr. B. Gärtner, Prof. J. Matoušek and S. Stich

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Approximation Algorithms and Semidefinite Programming FS12
Exercise Set 4

Course Webpage: `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP12/`
Discussion of 1 & 2: April 28; **Discussion of 3:** Mai 8

Exercise 1 (The Dual of the MaxCut SPD)

Let $G = (V, E)$ be a graph on n vertices, let $A = A_G$ be its adjacency matrix (i.e. the symmetric matrix with $a_{ij} = \delta_{\{i,j\}\in E}$) and let $\lambda_{\min}$ be the smallest eigenvalue of A. Recall that the MaxCut SDP was defined as follows:

$$\begin{array}{rll} \mathsf{SDP} = \text{Maximize} & \sum_{\{i,j\}\in E} \frac{1-x_{ij}}{2} & \\ \text{subject to} & x_{ii} = 1\,, & i = 1,\dots,n\,, \\ & X \succeq 0\,. & \end{array} \tag{1.3}$$

a) Derive the dual of (1.3).

b) Show that $\mathsf{SDP} \le \frac{1}{2}|E| + \frac{-\lambda_{\min} n}{4}$.

Exercise 2 (Integrality Gap of a Vertex Cover relaxation)

Let $G = (V, E)$ be a graph on n vertices. The goal of the *minimum vertex cover* problem is to find a minimum-size subset of vertices $C \subset V$ such that for every edge $\{i, j\} \in E$, $\{i, j\} \cap C \neq \{\varnothing\}$. This problem is equivalent to the following integer program:

$$\begin{array}{rll} \mathsf{OPT} = \text{Minimize} & \sum_{i\in V} x_i & \\ \text{subject to} & x_i + x_j \ge 1\,, & \forall\,\{i,j\} \in E\,, \\ & x_i \in \{0,1\}\,, & \forall\, i \in V\,. \end{array} \tag{2}$$

By replacing the constraints on the last row with $x_i \ge 0$ for $i \in V$ we get a linear program and we will denote its value by LP. Let $\mathsf{Gap} := \sup_G \frac{\mathsf{OPT(G)}}{\mathsf{LP(G)}}$ denote the integrality gap of this relaxation (note that this is a minimization problem).

a) Find a suitable family of graphs to show that $\mathsf{Gap} \ge 2 - \epsilon$ for any $\epsilon > 0$.

b) Let $\mathbf{y} \in \mathbb{R}^n$ be an optimal solution of the relaxation with value LP. Show that

$$x_i := \begin{cases} 1, & \text{if } y_i \ge \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

is a feasible solution to (2).

c) Use b) to show that the integrality gap is at most 2.

Exercise 3 (Integrality Gap of a Matching relaxation)

Let $G = (V, E)$ be a graph on n vertices. The goal of the *maximal matching* problem is to find a maximum-size set of edges $M \subset E$ such that no two edges of M share an endpoint. This problem is equivalent to the following integer program:

$$\begin{array}{rll} \mathsf{OPT} = \text{Maximize} & \sum_{e\in E} x_e & \\ \text{subject to} & \sum_{e\in\delta(i)} x_e \le 1\,, & \forall\, i \in V\,, \\ & x_e \in \{0,1\}\,, & \forall\, e \in E\,, \end{array} \tag{3}$$

where $\delta(i)$ denotes the set of edges incident to vertex i. By replacing the constraints on the last row with $x_e \geq 0$ for $e \in E$ we get the corresponding linear program relaxation and we will denote its value by LP. Let $\mathsf{Gap} := \sup_G \frac{\mathsf{LP(G)}}{\mathsf{OPT(G)}}$ denote the integrality gap of this relaxation. In this exercise we will show that $\mathsf{Gap} = \frac{3}{2}$.

a) Find an example graph that shows $\mathsf{Gap} \geq \frac{3}{2}$.

b) Let $\mathbf{y} \in \mathbb{R}^{|E|}$ be an optimal solution of the relaxation with value LP. Consider the graph $G' = (V, E')$ where $E' = \{e \in E \mid 0 < y_e < 1\}$. Assume that G' contains no cycle of even length.

- Show that G' consists of (edge) disjoint cycles of odd length.
- Conclude that $y_e = \frac{1}{2}$ for all $e \in E'$.
- Conclude that $\mathsf{Gap} \leq \frac{3}{2}$.

c) If G' contains a cycle of even length, show how to modify the solution $\mathbf{y}$ so that the objective value does not decrease and the number of edges in E' decreases by at least one.