# Menger's Theorem for directed graphs

Given $x, y \in V(D)$, a set $S \subseteq V(D) \setminus \{x, y\}$ is an $x, y$-separator (or an $x, y$-cut) if $D - S$ has no $x, y$-path.

Define

$\kappa_D(x, y)$ := min$\{|S|$ : $S$ is an $x, y$-cut,$\}$ and
$\lambda_D(x, y)$ := max$\{|\mathcal{P}|$ : $\mathcal{P}$ is a set of p.i.d. $x, y$-paths$\}$

**Directed-Local-Vertex-Menger Theorem** Let $x, y \in V(D)$, such that $\vec{xy} \notin E(D)$. Then

$$\kappa_D(x, y) = \lambda_D(x, y).$$

*Proof. (Aharoni)* Let $A = N^+(x)$ and $B = N^-(y)$.

$$\begin{aligned} D' := D - \{x, y\} \; & - \; \{\vec{za} : a \in A, z \in V(D)\} \\ & - \; \{\vec{bz} : b \in B, z \in V(D)\} \end{aligned}$$

$\mathcal{D}$: family of all $A, B$-paths in $D'$.

**GOAL:** Find a family $\mathcal{P} \subseteq \mathcal{D}$ of pairwise disjoint $A, B$-paths and a subset $S \subseteq V(D')$ such that
$|S \cap V(P)| \geq 1$ for every $P \in \mathcal{D}$ and
$|S \cap V(P)| = 1$ for every $P \in \mathcal{P}$.

Proving the GOAL is indeed enough. (Think it over)

*Proof of GOAL.* Define an auxiliary bipartite graph $H$.

$$\begin{aligned} V(H) &:= \{v^-, v^+ : v \in V(D')\} \\ E(H) &:= \{u^+v^- : uv \in E(D')\} \cup \\ &\quad\ \{v^-v^+ : v \in V(D') \setminus A \setminus B\} \end{aligned}$$

By König's Theorem there is a matching $M$ and a vertex-cover $C$ in $H$, such that $|e \cap C| = 1$ for every $e \in M$.

$$\mathcal{P} := \{x_1 \cdots x_k \in \mathcal{D} : x_i^+ x_{i+1}^- \in M \text{ for } 1 \leq i < k\}.$$
$$S := \{v \in V(D') \ : \ v^+, v^- \in C \text{ or } v^+ \in A^+ \cap C \text{ or } v^- \in B^- \cap C\}.$$

- Any two paths $P_1, P_2 \in \mathcal{P}$ are disjoint.

  $V(P_1) \cap V(P_2) \neq \emptyset$ implies there is $f_1 \in E(P_1)$, $f_2 \in E(P_2)$ such that $f_1 \neq f_2$ and $f_1 \cap f_2 \neq \emptyset$. $P_1, P_2 \in \mathcal{P}$ implies that for any $f_i \in E(P_i)$ either $f_1 = f_2$ or $f_1 \cap f_2 = \emptyset$.

- Any $A, B$-path $x_0 x_1 x_2 \cdots x_k$ contains a vertex from $S$.

  Let $i$ be the largest index such that $x_i^- \in C$. (There is such, unless $x_0^+ \in C$ and $i < k$ unless $x_k^- \in C$)
  Then $x_i^+ \in C$ since $x_i^+ x_{i+1}^-$ must be covered.

- No $A, B$-path $u_0 u_1 u_2 \cdots u_k = P \in \mathcal{P}$ contains more than one vertices from $S$.

  Suppose $P$ does contain more. Let $u_i$ and $u_j \in S \cap V(P)$ such that $u_k \notin S$ for $i < k < j$. Then $u_i^+, u_j^- \in C$ by definition of $S$. Let $k$ be the largest index, $i < k < j$, such that $u_k^+ \in C$. Then $u_{k+1}^- \in C$ to cover the edge $u_{k+1}^- u_{k+1}^+$. Hence edge $u_k^+ u_{k+1}^- \in M$ is covered twice by $C$, a contradiction.

## Corollaries

**Corollary** (Directed-Global-Vertex-Menger Theorem) A digraph $D$ is strongly $k$-connected iff for any two vertices $x, y \in V(D)$ there exist $k$ p.i.d. $x, y$-paths.

*Proof:* **Lemma.** For every $e \in E(D)$, $\kappa_D(G-e) \geq \kappa_D(G)-1$.

The proof of the very first, the original Menger Theorem (the Undirected-Local-Vertex version) is

**HOMEWORK !!!**

Derive implication DLVM $\Rightarrow$ ULVM

## Directed Edge-Menger

Given $x, y \in V(D)$, a set $F \subseteq E(D)$ is an $x, y$-disconnecting set if $D - F$ has no $x, y$-path. Define

$\kappa'_D(x, y)$ := min{$|F|$ : $F$ is an $x, y$-disconnecting set,}
$\lambda'_D(x, y)$ := max{$|\mathcal{P}|$ : $\mathcal{P}$ is a set of p.e.d.* $x, y$-paths}

* p.e.d. means pairwise edge-disjoint

**Directed-Local-Edge-Menger Theorem** For all $x, y \in V(D)$,

$$\kappa'_D(x, y) = \lambda'_D(x, y).$$

*Proof.* Create directed line graph and apply DLVM.

**Corollary** (Directed-Global-Edge-Menger Theorem) Directed multigraph $D$ is strongly $k$-edge-connected iff there is a set of $k$ p.e.d. $x, y$-paths for any two vertices $x$ and $y$.