Covering Polygons with Few Rectangles
(Extended Abstract)

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Abstract

How to cover a set of planar polygons with two or three congruent axis-parallel squares of minimal size? I show that this natural generalization of the bounding-box can be computed in time linear in the number of vertices.

1 Introduction

Consider the following problem for illustration. You are given a map of some country, state or city and want to distribute it among the pages of an atlas such that the scale is maximized. Assuming the map consists of a polygonal region and the p pages of the atlas have rectangular shape and are of the same size and given orientation, the problem can be rephrased more precisely as follows.

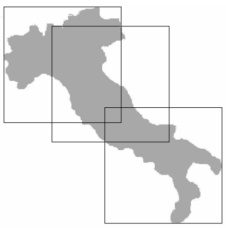

Figure 1: Example Covering for $p = 3$.

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Given a set $\mathcal{P}$ of pairwise non-intersecting polygonal regions and a number $p \in \mathbb{N}$, find p axis-parallel congruent squares of minimal size covering $\mathcal{P}$. Covering means here that any point from any polygon in $\mathcal{P}$ is contained in at least one of the p squares. Note that replacing rectangles of given aspect ratio by squares is just a matter of an appropriate scaling of one coordinate axis. The resulting set of p squares will be called (a) minimal (rectilinear) p-covering of $\mathcal{P}$ and can be seen as a natural generalization of the bounding box of $\mathcal{P}$.

If the input consists of points instead of polygons, the problem is known as rectilinear p-center problem [4, 13]. So, another way to look at polygon covering is as a generalization of the rectilinear p-center problem to infinite point sets. Since the latter problem cannot be approximated within a factor of less than two unless $P = NP$ [10, 9], one cannot expect to do better for the more general problem of covering polygons.

Despite the apparent intractability, efficient algorithms have been developed for small values of p. The rectilinear 2– and 3-center problem can be solved in linear time [4, 8, 13], while for $p = 4$ already there is a lower bound of $\Omega(n \cdot \log n)$ and algorithms of matching complexity for $p \leq 5$ [3, 11, 12]. The hope is that also the polygon covering problem can be solved efficiently for a small number of covering squares.

2 Results

While in principle techniques similar to the ones used for covering points can also be applied to cover polygons, there is one important difference between both problems that has to be addressed: In the point problem, the size of an optimal covering is always determined by the L_∞-distance of two points. Hence, by using sorted matrix search [5, 6, 7] the complexity of the decision problem, whether a covering smaller than a certain value exists, is within a log factor of the optimization problem for any number of covering squares. But this argument does not hold for the case of polygons!

Nevertheless, for $p = 2$, polygon, or more generally, line segment covering can be reduced to point covering. It is easy to show that in this case, covering a line segment is equivalent to cover both its endpoints and a another reference point that is determined by the segment and the bounding box of the whole input set only. From the known results on point covering [4], we can conclude the following.

Theorem 1 *A minimal rectilinear 2-covering of a given set of line segments in the plane can be computed in linear time.*

For $p \leq 3$ it is no restriction to focus on line segments only: it is sufficient to cover the boundary edges of the polygons, as the union of three axis-parallel rectangles in 2D has always genus zero.

Corollary 2 *A minimal rectilinear 2-covering of a given set of polygonal regions in the plane can be computed in time linear in the number of vertices.*

For $p \geq 3$ however, there does not seem to be an easy way to reduce line segment covering to point covering. Hence, I took a different approach which lead to the development of another matrix search technique. This technique can be applied to compute 3-coverings of points, line segments and/or polygons based on a purely combinatorial description.

Theorem 3 *A minimal rectilinear 3-covering of a given set of line segments in the plane can be computed in linear time.*

Corollary 4 *A minimal rectilinear 3-covering of a given set of polygonal planar regions with n vertices in total can be computed in $\mathcal{O}(n)$ time.*

This result might be a bit surprising, considering the fact that the planar arrangement of a set of line segments has quadratic complexity in general and even a single cell of it can have super-linear complexity [1].

While the techniques in principle generalize to higher dimensions, the combinatorics of the covering boxes changes, even if the input consists of points only. Already for $p = 3$ in dimension three, it is e.g. no longer true that one of the covering boxes has to be placed at a corner of the overall bounding box; a minimal covering can also consist of boxes placed at the interior of three opposite edges of the bounding box, as shown in Figure 2.

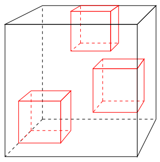

Figure 2: Three-Covering with no box at a corner.

Indeed, these configurations turn out to be the difficult ones, giving raise to the following lower bound, matching an algorithm of Assa and Katz [2].

Theorem 5 *Computing a minimal rectilinear three-covering of n points in $\mathbb{R}^3$ requires $\Omega(n \cdot \log n)$ operations in the algebraic computation tree model.*

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