A Multidimensional Generalization of the Erdős-Szekeres Lemma on Monotone Subsequences

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Abstract

We consider an extension of the Monotone Subsequence Lemma of Erdős and Szekeres in higher dimensions. Let $v^1, \ldots v^n \in \mathbb{R}^d$ be a sequence of real vectors. For a subset $I \subseteq [n]$ and vector $\vec{c} \in \{0,1\}^d$ we say that I is $\vec{c}$-free, if there are no $i < j \in I$, such that for every $k = 1, \ldots d$, $v_k^i < v_k^j$ iff $\vec{c}_k = 0$. We construct sequences of vectors with the property, that the largest $\vec{c}$-free subset is *small* for every choice of $\vec{c}$. In particular, for $d = 2$ the largest $\vec{c}$-free subset is $O(n^{\frac{5}{8}})$ for all the four possible $\vec{c}$. The smallest possible value remains far from being determined.

We also consider and resolve a simpler variant of the problem.

1 Introduction

The classic lemma of Pál Erdős and György Szekeres [5] about monotone subsequences can also be formulated as a Ramsey-type coloring statement.

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Lemma 1 *[5] Let H_0 and H_1 be linear orderings of the n-element set V. Define a 2-coloring of the edges of the complete graph on vertex set V by coloring the edge uv blue if the order of u and v in H_0 agrees with that in H_1, and coloring it red otherwise. Then there exists a monochromatic clique of size $\lceil \sqrt{n} \rceil$.*

Moreover, this result is best possible. That is, there exist linear orderings H_0, H_1 such that in the corresponding coloring the largest monochromatic clique is of size $\lceil \sqrt{n} \rceil$.

In the present paper we consider a generalization of this Lemma.

Let $\mathcal{H} = (H_0, \ldots, H_d)$ be a list of $d+1$ linear orderings on a finite set V. Let us 2^d-color the edges of the complete graph on the vertex set V by coloring the edge uv with the color $(c_1, \ldots, c_d) \in \{0,1\}^d$, where $c_i = 0$ if H_i agrees with H_0 on $\{u, v\}$, and $c_i = 1$ otherwise.

The first natural generalization of Lemma 1 coming into mind is about determining the size of the largest monochromatic subset one can guarantee. As it turns out this question is solved easily with repeated applications of the Erdős-Szekeres Lemma (N. G. de Bruijn, see [7]). Instead, we concentrate on the property of a monochromatic subset in a 2-edge-coloring, that it does *not* contain *all* the colors. We will try to determine the size of the largest subset missing at least one of the 2^d colors.

We found this problem interesting in its own combinatorial right. The original motivation, however, stems from real analysis. M. Laczkovich [8] raised the problem whether any compact set of positive Lebesgue measure in d-space admits a contraction onto a ball. J. Matoušek [11] formulated a related combinatorial question; we consider yet a slightly different version with implications to the original analysis problem of Laczkovich.

We note here that several other extensions of Lemma 1 exist in the literature. The paper of V. Chvátal and J. Komlós [4] generalizes the transitive tournament structure of linear orderings to arbitrary directed graphs. In a more recent paper, R. Siders [12] considers a version in higher dimensions. He resolves the question of M. Kruskal (see [7]) about how small the size of the largest monotone subsequence could be in *any* direction (not just in the direction of the coordinate axes).

Let us make things more precise by introducing a few definitions. For $\vec{c} \in \{0,1\}^d$ we call a subset $U \subseteq V$ *$\vec{c}$-free* if U spans a subgraph with no edge of color $\vec{c}$. We define $m_{\vec{c}}(V, \mathcal{H})$ to be the size of the maximal $\vec{c}$-free subset in V. Let $m(V, \mathcal{H}) = \max_{\vec{c}} m_{\vec{c}}(V, \mathcal{H})$ and $m(n, d) = \min m(V, \mathcal{H})$ where the maximum ranges over the colors $\vec{c} \in \{0,1\}^d$ and the minimum ranges over the n element sets V and the lists $\mathcal{H}$ of $(d+1)$ linear orderings of V.

PROBLEM: Determine $m(n, d)$. Find the order of magnitude for fixed $d \geq 0$.

All orders of magnitude, and all the O, o, and Θ notations in this paper are in the variable n with respect to a fixed d unless otherwise stated.

We trivially have $m(n,0) = 1$. For $d = 1$ our problem reduces to the lemma of Erdős and Szekeres and one gets $m(n,1) = \lceil\sqrt{n}\,\rceil$. For $d > 1$, however, the problem starts to get interesting. A trivial lower bound is $m(n,d) \geq m(n,1) = \lceil\sqrt{n}\,\rceil$ for $d \geq 1$.

For an easy upper bound of $m(n,d) = O(n^{\frac{d}{d+1}})$, one can generalize the construction usually associated with the second part of Lemma 1.

Construction 1

Let $n = n_0^{d+1}$, let V consist of the $(d+1)$-tuples from $\{0,\dots,n_0-1\}$ let $\mathcal{H} = (H_0,\dots,H_d)$ such that H_i extends the natural ordering according to the i^{th} coordinate (label the coordinates from 0 to d). Let $\vec{c} \in \{0,1\}^d$ be a color. We define a partition of V into at most $(3n_0-2)^d$ monochromatic subsets $R_{\vec{a}}$, where $\vec{a} \in \{-(n_0-1),\dots,0,1,\dots 2n_0-2\}^d$. Let

$$R_{\vec{a}} = \{(x_0,x_1,\dots,x_d) \in V : \vec{a} = (x_1,\dots,x_d) + x_0(2\vec{c} - (1,\dots,1))\}.$$

It is clear that each $R_{\vec{a}}$ is monochromatic in the color $\vec{c}$, thus a $\vec{c}$-free subset does not contain more than one element of it. This implies $m_{\vec{c}}(V,\mathcal{H}) \leq (3n_0-2)^d$ and since $\vec{c}$ was arbitrary we have $m(n,d) \leq m(V,\mathcal{H}) \leq (3n_0-2)^d = O(n^{\frac{d}{d+1}})$. $\Box$

The problem of obtaining a decent lower bound resisted our attempts so far. Currently we do not know anything better than $m(n,d) \geq \sqrt{n}$. This is immediate from the Erdős-Szekeres Lemma and proves unnecessarily too much. It provides a subset of size $\lceil\sqrt{n}\,\rceil$ free of not just one, but half of the colors.

With the hope that it might shed some light on the problem, L. Pósa (see in [9]) suggested a related simpler question. Instead of $d+1$ linear orderings, consider a d-tuple $\mathcal{P} = (\mathcal{P}_1,\dots,\mathcal{P}_d)$ of partitions of a base set V. With the aid of these partitions, we define a 2^d-edge-coloring of the complete graph on vertex set V by letting $\vec{c} = (c_1,c_2,\dots,c_d) \in \{0,1\}^d$ be the color of the edge uv, where $c_i = 0$ if u and v are in the same class of $\mathcal{P}_i$ and $c_i = 1$ otherwise. We define the analogue of $m_{\vec{c}}(V,\mathcal{H}), m(V,\mathcal{H})$, and $m(n,d)$ for this coloring. For $\vec{c} \in \{0,1\}^d$ let $r_{\vec{c}}(V,\mathcal{P})$ be the size of the maximal $\vec{c}$-free subset of V. Let $r(V,\mathcal{P}) = \max_{\vec{c}} r_{\vec{c}}(V,\mathcal{P})$, and $r(n,d) = \min r(V,\mathcal{P})$, where the maximum ranges over the colors $\vec{c} \in \{0,1\}^d$ and the minimum ranges over the n element sets V and the d-tuples $\mathcal{P}$ of partitions of V. Our goal is also similar: the determination of $r(n,d)$.

In Section 2 we consider this problem on partitions. As it turns out Construction 1 can be translated into the language of partitions. We also prove a matching lower bound in Theorem 2, thus obtain a precise answer for many values of the parameter n: $r(n^{d+1},d) = n^d$.

This implies the asymptotic characterization of $r(n,d)$: $r(n,d) \approx n^{\frac{d}{d+1}}$. The same result was independently proved by L. Pósa (for $d=2$), Gy. Petruska [9] and it is also a consequence of a result of A. Schwenk and I. Munro [13]. Our approach provides a more general statement in a different sense.

One could also consider the random version of the original problem. Brightwell [3, Corollary 2], answering a problem raised by Winkler [14], showed that Construction 1 is very typical in the following sense. If we choose $d+1$ linear orderings independently and uniformly from all linear orderings of the set V, then almost always $m(V,\mathcal{H}) = \Theta(n^{\frac{d}{d+1}})$.

After seeing the results of the simpler variant and the random version, it might seem plausible to conjecture for our original problem that $m(n,d) = \Theta(n^{\frac{d}{d+1}})$. As we show in Section 3, this, however, is *not* the case. We improve on Construction 1 to obtain $m(n,d) = O(n^{e_d})$, with an exponent satisfying $e_d < \frac{d}{d+1}$ for $d \geq 2$. For example we have $m(n,2) = O(n^{5/8}) \ll n^{2/3}$. For large values of d we have $e_d = 1 - 2/d + o(1/d)$, where the o bound is in the variable d.

2 Partitions

Theorem 2 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_d$ be d partitions of the n element set V. Let us define a 2^d-edge-coloring of the complete graph on vertex set V by letting $\vec{c} = (c_1, c_2, \ldots, c_d) \in \{0,1\}^d$ to be the color of the edge uv, where $c_i = 0$ if u and v are in the same class of $\mathcal{P}_i$ and $c_i = 1$ otherwise.*

There exists a color $\vec{c}$ and a $\vec{c}$-free subset $B \subseteq V$, with $|B| \geq n^{\frac{d}{d+1}}$.

We obtain Theorem 2 as a consequence of a stronger statement. We consider a coarser coloring of K_n, by just $d+1$ colors, and prove that the geometric average of the sizes of the largest subsets avoiding one of these colors is at least $n^{d/(d+1)}$.

Theorem 3 *Let $\mathcal{P}_1, \ldots, \mathcal{P}_d$ be d partitions of the n element set V. Let us define a $d+1$-edge-coloring of the complete graph on vertex set V by coloring the edge uv by the largest value i ($0 \leq i \leq d$) such that u and v are in the same class of the first i partitions.*

Then there exist subsets $B_0, B_1, \ldots, B_d$ of V, where B_i spans a subgraph free of color i, and

$$\prod_{i=0}^{d} |B_i| \geq n^d.$$

Notice that any i-free subset of V is free of not just one, but 2^{d-i} colors in the coloring of Theorem 2. Hence Theorem 2 follows as a consequence.

Proof of Theorem 3. We proceed by induction on d. The case $d=1$ is immediate. Now let us assume that the statement is true for $d-1$.

Consider the restrictions of $\mathcal{P}_2, \ldots, \mathcal{P}_d$ to the classes $S_1, \ldots, S_k$ of $\mathcal{P}_1$. By the induction hypothesis there exist subsets $B_1^j, \ldots, B_d^j$ of S_j for every j $(1 \le j \le k)$, such that $\prod\limits_{i=1}^{d} |B_i^j| \ge |S_j|^{d-1}$ and B_i^j is i-free. We define $B_i := \cup_{j=1}^k B_i^j \subseteq V$ for $1 \le i \le d$. B_i is clearly an i-free set.

As $\sum_{j=1}^k |S_j| = n$, we can use the weighted version of the inequality between the arithmetic and geometric mean, with weights $|S_j|/n$, to estimate the size of B_i.

$$|B_i| = \sum_{j=1}^{k} |B_i^j| = n \sum_{j=1}^{k} \frac{|S_j|}{n} \frac{|B_i^j|}{|S_j|} \ge n \prod_{j=1}^{k} \left(\frac{|B_i^j|}{|S_j|} \right)^{\frac{|S_j|}{n}}. \tag{1}$$

Let us define B_0 to be one of the largest classes in $\mathcal{P}_1$ and let $t = |B_0|$. B_0 is clearly 0-free. Using (1) and the induction hypothesis, we obtain

$$\prod_{i=0}^{d} |B_i| \ge t \prod_{i=1}^{d} n \prod_{j=1}^{k} \left(\frac{|B_i^j|}{|S_j|} \right)^{\frac{|S_j|}{n}} = tn^d \frac{\prod_{j=1}^k (\prod_{i=1}^d |B_i^j|)^{\frac{|S_j|}{n}}}{(\prod_{j=1}^k |S_j|^{\frac{|S_j|}{n}})^d} \ge$$

$$\ge tn^d \frac{\prod_{j=1}^k |S_j|^{\frac{(d-1)|S_j|}{n}}}{\prod_{j=1}^k |S_j|^{\frac{d|S_j|}{n}}} = n^d \frac{t}{\prod_{j=1}^k |S_j|^{\frac{|S_j|}{n}}} \ge n^d.$$

In the last inequality the denominator is a weighted geometric average of the $|S_j|$, whereas the numerator is their maximum, so Theorem 3 follows. □

Recall the definition of $r(n,d)$ from the Introduction.

Corollary 4 *For positive integers d and n_0 we have*

$$r(n_0^{d+1}, d) = n_0^d.$$

For arbitrary n and fixed d we have

$$r(n,d) = (1+o(1)) n^{\frac{d}{d+1}}.$$

Proof. Theorem 2 provides the lower bound on $r(n,d)$.

Construction 1 can be transformed into a construction of partitions and provides the upper bound. Let $n = n_0^{d+1}$, and $V = \{0, 1, \ldots, n_0 - 1\}^{d+1}$. For each $i = 1, 2, \ldots, d$ we define the partition $\mathcal{P}_i$ using the i^{th} coordinates: two elements of V are in the same class of $\mathcal{P}_i$ if they have the same i^{th} coordinate. (Coordinates are labeled from 0 to d.)

Take a color $\vec{c} = (c_1, \ldots, c_d)$. One can partition V into $\vec{c}$-monochromatic subsets $R_{\vec{a}}$ of size n_0 ($\vec{a} \in \{0, 1, \ldots n_0 - 1\}^d$). Indeed, let

$$R_{\vec{a}} = \{(i, \vec{x}) : i \in \{0, 1 \ldots, n_0 - 1\}, \vec{x} = \vec{a} + i\vec{c}\},$$

where the sum in the coordinates is computed modulo n_0.

Thus for any color $\vec{c}$, the size of the largest $\vec{c}$-free subset is at most n_0^d. $\Box$

Remark Theorem 2 was also proved independently by Pósa for $d = 2$, and Petruska [9] for arbitrary d. Their proof is different from ours.

A. Schwenk and I. Munro [13] found a generalization of Theorem 2 in a direction differing from the one of Theorem 3. They showed, that for any $1 \leq k \leq l$ and any subset $V \subseteq \mathbb{R}^l$ of size n, the geometric mean of the sizes of the projections to the $\binom{l}{k}$ possible k-dimensional subspaces spanned by coordinate axes is at least $n^{k/l}$. Applied with $l = d+1, k = d$, this result is equivalent to the following statement in the setting of Theorem 2: the geometric mean of the sizes of the largest subsets of V, free of $d+1$ specific colors (the colors having at most a single 1), is at least $n^{\frac{d}{d+1}}$. This is also a special case of our Theorem 3.

3 The construction

In this section we present a generalization of Construction 1 to improve on the exponent of the upper bound for $m(n, d)$ provided $d \geq 2$.

Theorem 5 *We have $m(n, d) = O(n^{e_d})$, with*

$$e_d = 1 - \max_{i \leq d+1} \frac{\sum_{j=0}^{i-1} \binom{d}{j}}{i 2^d}.$$

In particular, $m(n, 2) = O(n^{5/8})$ and there exists an absolute constant $c > 0$, such that for every fixed d we have

$$m(n, d) < O\left(n^{1 - \frac{2}{d} + \frac{c\sqrt{\log d}}{d^{3/2}}}\right).$$

Proof. We start by defining a product operation. Let $\mathcal{H}' = (H'_0, \ldots, H'_d)$ be $d+1$ linear orderings on the finite set V' and let $\mathcal{H}'' = (H''_0, \ldots, H''_d)$ be $d+1$ linear orderings on the finite set V''. We define $V = V' \times V''$ to be the Cartesian product and $\mathcal{H}' \times \mathcal{H}'' = \mathcal{H} = (H_0, \ldots, H_d)$ to be $d+1$ orderings on V, where H_i is the lexicographic ordering of V using the orderings H'_i and H''_i on the coordinates ($i = 0, 1, \ldots, d$).

Lemma 6 *For any color $\vec{c} \in \{0, 1\}^d$ we have $m_{\vec{c}}(V, \mathcal{H}) = m_{\vec{c}}(V', \mathcal{H}') \cdot m_{\vec{c}}(V'', \mathcal{H}'')$.*

Proof. For the $\geq$ direction take subsets S' of V', and S'' of V'', which are both $\vec{c}$-free and notice that $S' \times S'' \subseteq V$ is also $\vec{c}$-free.

For the $\leq$ direction of the claim take a $\vec{c}$-free subset $S \subseteq V$. Note that the projection S' of S to V' is $\vec{c}$-free, and the slice $S_a = \{b \in V'' | (a,b) \in S\}$ is also $\vec{c}$-free for any $a \in V'$. $\square$

In what follows, we modify Construction 1 to obtain a list $\mathcal{H} = (H_0, \ldots, H_d)$ of linear orderings such that $m_{\vec{c}}(V, \mathcal{H})$ is substantially lower for most of the colors but it is very high (in fact n) for the remaining colors. Then we use the product construction of Lemma 6 for averaging.

Let $1 \leq i \leq d+1$ and let us take an i dimensional linear subspace W of $\mathbb{R}^{d+1}$ in general position with respect to the coordinate axes. In the following we consider d, i and W fixed, and use the O notation with respect to n. Consider (a rotation of) the i dimensional unit square grid in W. Let V be the n points of this grid closest to the origin. Notice that the diameter of V is $O(n^{1/i})$. For $i = 0, 1 \ldots, d$ define H_i to be the linear ordering given by the ordering of the i^{th} coordinates of the points. (Coordinates are labeled from 0 to d.)

The color of the edge $\vec{u}\vec{v}$ ($\vec{u}, \vec{v} \in V$) depends on which of the 2^{d+1} space orthant contains the vector $\vec{u} - \vec{v}$. An orthant Q is associated with the same color as $-Q$, hence the $2^{d+1}/2$ colors. Let us denote by $Q_{\vec{c}}$ the union of the two orthants associated with the color $\vec{c}$. In the following claim we show that the magnitude of $m_{\vec{c}}(V, \mathcal{H})$ depends only on whether $Q_{\vec{c}} \cap W$ is trivial or not.

Claim 7 *If $Q_{\vec{c}} \cap W = \{0\}$ then $m_{\vec{c}}(V, \mathcal{H}) = n$. If $Q_{\vec{c}} \cap W \neq \{0\}$, then $m_{\vec{c}}(V, \mathcal{H}) = O(n^{\frac{i-1}{i}})$.*

Proof. The first statement simply follows from the definition of $Q_{\vec{c}}$ and from the fact that W is closed under subtraction.

To prove the second statement choose a vector $\vec{v} \in W$ that lies in the interior of $Q_{\vec{c}}$. Let $S \subseteq V$ be a subset not containing the color $\vec{c}$.

Let us project S in the direction of $\vec{v}$ onto the subspace of W orthogonal to $\vec{v}$ and call the projected point set S_0. Recall that $\vec{v}$ is in the interior of $Q_{\vec{c}}$ and thus there exists a positive angle α with the property that any vector within angle at most α from $\vec{v}$ is in $Q_{\vec{c}}$. Clearly for any two vectors $\vec{u}$ and $\vec{w} \in S$ the distance of their projections $\vec{u_0}, \vec{w_0} \in S_0$ is at least $\sin \alpha$ times the distance of $\vec{u}$ and $\vec{w}$, as otherwise the difference $\vec{u} - \vec{w}$ (or $\vec{w} - \vec{u}$) is within angle α from $\vec{v}$, making $\vec{c}$ the color of the edge $\vec{u}\vec{v}$, a contradiction. Thus the minimum distance in S_0 is constant, while the diameter is at most the diameter of S, which is $O(n^{1/i})$. A simple volume calculation shows that $|S| = |S_0| = O(n^{\frac{i-1}{i}})$. $\square$

Suppose that in the construction above we have $l = l(i)$ colors with $m_{\vec{c}}(V, \mathcal{H}) = O(n^{1-1/i})$ and $2^d - l$ colors with $m_{\vec{c}}(V, \mathcal{H}) = n$. There are 2^d ways to reverse some of the linear orderings

$H_1, \ldots, H_d$ and obtain a different construction $\mathcal{H}^j$, $j = 1, 2, \ldots, 2^d$. For each of them the set of l colors with $m_{\vec{c}}(V, \mathcal{H}^j) = O(n^{\frac{i-1}{i}})$ might be different. Because of symmetry any fixed color $\vec{c}$ occurs l times out of the 2^d with $m_{\vec{c}}(V, \mathcal{H}^j) = O(n^{1-1/i})$.

Now we use the product construction to multiply these 2^d systems. For the resulting family $(V^*, \mathcal{H}^*)$ the values $m_{\vec{c}}(V^*, \mathcal{H}^*)$ average out for every color $\vec{c}$,

$$m_{\vec{c}}(V^*, \mathcal{H}^*) = \prod_{j=1}^{2^d} m_{\vec{c}}(V, \mathcal{H}^j) = O\left(n^{l\frac{i-1}{i}+(2^d-l)}\right) = O\left(N^{1-\frac{l}{i2^d}}\right),$$

where $N = |V^*| = n^{2^d}$

Assuming the next lemma on the value of l (as a function of i) the first statement of Theorem 5 follows. For the last statement of the theorem notice that with the choice $i = \lfloor d/2 + 10\sqrt{d \log d} \rfloor$ Chernoff bound gives $\sum_{j=0}^{i-1} \binom{d}{j}/(i2^d) = 2/d - O(\sqrt{\log d/d^3})$ where the O is with respect to d. $\square$

Lemma 8 *If $W \subseteq \mathbb{R}^{d+1}$ is an i-dimensional linear subspace in general position with respect to the coordinate axes, then W nontrivially intersects exactly $2\sum_{j=0}^{i-1} \binom{d}{j}$ of the 2^{d+1} orthants of $\mathbb{R}^{d+1}$.*

Proof. The intersections of the $d+1$ coordinate hyperplanes of $\mathbb{R}^{d+1}$ with W are $d+1$ $(i-1)$-dimensional linear subspaces of W in general position. Thus counting the $(d+1)$-dimensional orthants intersected nontrivially by W is the same as counting the connected parts $\mathbb{R}^i$ is cut by $d+1$ subspaces of dimension $i-1$ in general position.

Our formula for this number can easily be established by a recurrence relation. We tried to find the oldest reference instead. In 1852 L. Schläfli [10] proved that j affine hyperplanes in general position partition the Euclidean k-space into $a(j,k) = \sum_{t=0}^{k} \binom{j}{t}$ parts. Notice that he uses affine subspaces and we use linear subspaces. We partition the i-space by $d+1$ linear subspaces of dimension $i-1$. Fix one of the subspaces S and consider affine hyperplanes S_1 and S_2 parallel to S that lie on different sides of S. Clearly each part of the i-space intersects exactly one of S_1 or S_2, thus the number of parts in our partition is the total number of parts S_1 and S_2 is partitioned by the other d of our subspaces. As the subspaces different from S intersect S_1 and S_2 in affine subspaces in general position we have that both S_1 and S_2 is partitioned into $a(i-1,d)$ parts, proving the theorem. $\square$

In Theorem 5 the value of e_d is defined with the help of the dimension parameter i. The construction in the proof works for each value $1 \le i \le d+1$ and choosing i optimally yields the exponent e_d. Observe that the choice $i = d+1$ provides a version of Construction 1 from the Introduction. By choosing $i = d$ we obtain a construction beating the random one even for small values of d. For $d = 2$, 3, and 4 this is the optimal choice for i and we get $m(n,2) = O(n^{5/8})$, $m(n,3) = O(n^{17/24})$, and $m(n,4) = O(n^{49/64})$. For $d = 5$, the optimal choice is $i = 4$ that yields $m(n,5) = O(n^{51/64})$. For large values of d the optimal choice for i is $d/2 + O(\sqrt{d \log d})$ yielding $e_d = 1 - \frac{2}{d} + O\left(\frac{\sqrt{\log d}}{d^{3/2}}\right)$, where the O notation refers to asymptotics in d.

4 Concluding Remarks and Open Problems

Remarks.

1. It would seem natural to try to prove an $n^{\frac{d}{d+1}}$ lower bound to the average of the sizes $|B_{\vec{c}}|$ for *all* 2^d colors $\vec{c}$ in the setting of Theorem 2, where $B_{\vec{c}} \subseteq V$ is a $\vec{c}$-free subset of maximum size. This is not possible because there are counterexamples with two partitions, where $\sum_{i=1}^{4} |B_i| < 4n^{2/3}$. Theorem 3 implies this bound for many subsets of the set of colors and it would be interesting to characterize the subsets of colors, for which such a bound holds.
2. Our argument for Theorem 3 gives a similar result for a coloring induced by 2 linear orderings and $d - 1$ partitions.
3. Since the proof of the current best lower bound for $m(n,2)$ provides a subset of size $n^{1/2}$ containing only 2 of the 4 colors, it seems reasonable to conjecture, that $\lim_{n\to\infty} m(n,2)/n^{1/2} = \infty$. We don't know anything better for large d either. As a first step, it would even be interesting to see whether there is a constant d, for which $\lim_{n\to\infty} m(n,d)/n^{1/2} = \infty$.
4. The result of Brightwell [3, Corollary 2] states that if $d + 1$ linear orders are chosen independently and uniformly out of all linear orders of V, then almost always (i.e. with probability tending to 1)

$$C_1 n^{\frac{d}{d+1}} < m(V, \mathcal{H}) < C_2 n^{\frac{d}{d+1}}.$$

This also implies that the median of $m(V, \mathcal{H})$ is $\Theta(n^{\frac{d}{d+1}})$. Using standard technic involving Talagrand's Inequality, one can obtain a stronger concentration inequality about the value of $m(V, \mathcal{H})$.

Theorem 9 *Let m be the median of $m(V, \mathcal{H})$ and let $\omega(n) \to \infty$ arbitrarily slowly. Then*

$$Pr(|m(V, \mathcal{H}) - m| > \omega(n) n^{\frac{1}{2} - \frac{1}{2(d+1)}}) = o(1).$$

The proof is a standard application of Talagrand's Inequality. For a clear explanation of this powerful probabilistic tool, see for example [1]. The proof of the case $d = 1$ is included there [1, page 109], and the exact same argument works word by word for arbitrary d. We don't include the details here.

For the case $d = 1$, Baik, Deift and Johansson [2] obtained that the actual variation of $m(V, \mathcal{H})$ about its median is of the order $n^{1/6}$ (and not $n^{1/4}$, given by the Talagrand inequality). They actually established a very precise estimate of the distribution of $m(V, \mathcal{H})$. It would be interesting to find a precise estimate on the distribution (or at least the variation) of $m(V, \mathcal{H})$ in higher dimension as well.

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