Testing Extendability for Partial Chirotopes is NP-Complete

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Abstract

This paper shows that it is NP-complete to algorithmically decide whether there exists a chirotope which extends a given partial chirotope.

While Crippen and Havel ([3]) gave an algorithm to compute all possible extensions of a given partial chirotope running in exponential time, Gärtner et al. ([6]) discussed a special class of partial chirotopes which turned out to be always extendable.

1 (Partial) Chirotopes

Oriented matroids are fundamental objects in combinatorial geometry which appear in many disguises. In particular, they can be considered as a combinatorial model of the determinant values on vector configurations.

Given a configuration $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ over $\mathbb{R}^r$, where r is called the *rank*, we consider the signs of the determinants of the ordered r-subsets of E. This motivates to define the *chirotope* of a vector configuration by the function $\chi(i_1, \ldots, i_r) := \text{sign}\det(\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_r}) \in \{+, -, 0\}$, identifying the vectors by their indices. Note that χ is an alternating sign map, i.e. for any permutation $\pi \in \Sigma_r$ we have $\chi(\pi(i_1), \pi(i_2), \ldots, \pi(i_r)) = \text{sgn}(\pi)\chi(i_1, \ldots, i_r)$. Furthermore, it satisfies the so-called Grassmann-Plücker relations. So, e.g. for rank $r = 3$, reduced to the sign patterns they imply, we get the following condition: for any five elements $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_5} \in E$ the set $\{\chi(i_1,i_2,i_3)\chi(i_1,i_4,i_5), -\chi(i_1,i_2,i_4)\chi(i_1,i_3,i_5), \chi(i_1,i_2,i_5)\chi(i_1,i_3,i_4)\}$ either contains $\{+1, -1\}$ or equals $\{0\}$.

Reversely, given a set E and $r \in \mathbb{N}$ with $3 \leq r \leq |E|$ an alternating sign map $\chi : E^r \rightarrow \{-1, 0, 1\}$ defines a chirotope of rank r on E whenever it satisfies the conditions mentioned above. (Cf. [1], [2].)

If χ is as above but only defined partially, it defines a *partial chirotope*. (For a precise definition, see e.g. Definition 1.12 in [2].)

Finally, we call a partial chirotope χ' of rank r on a set E *extendable*, if there exists a chirotope χ of rank r on E and for any $x_1, \ldots, x_r \in E$, $\chi(x_1, \ldots, x_r) = \chi'(x_1, \ldots, x_r)$ holds whenever $\chi'(x_1, \ldots, x_r)$ is defined.

We are now ready to state the problem we are going to discuss:

Problem 1.1 *Given a partial chirotope χ' of rank r on a set E, how hard is it to decide algorithmically that it is extendable?*

Throughout this paper we will deal only with (partial) uniform chirotopes of rank 3, that is, 0 is not in the image of χ. By the Folkman-Lawrence representation theorem [4] every oriented matroid of rank 3 can be identified as an arrangement of oriented pseudolines. This is a collection of simple closed curves in the real projective plane where any two intersect in exactly one point. Uniformity now means that no three pseudolines meet in a common point. We can single out a particular pseudoline that can be identified with the line at infinity of the usual Euclidean plane.

If we are given such a pseudoline arrangement, it is simple to read off the chirotope: Any three pseudolines i, j, k, say, enclose a pseudotriangle. Denoting by n the number of lines pointing *inwards*, we define $\chi(i, j, k)$ by $(-1)^{n+1}$ and $(-1)^n$, resp., whenever i, j, k bound the pseudotriangle in clockwise and counterclockwise order, resp. Figure 1 demonstrates how given two pseudolines ℓ, ℓ' the orientation and location of a third pseudoline p effects the value of $\chi(\ell, \ell', p)$.[1] It shall serve as an easy reference point for the reader when studying more complicated drawings later.

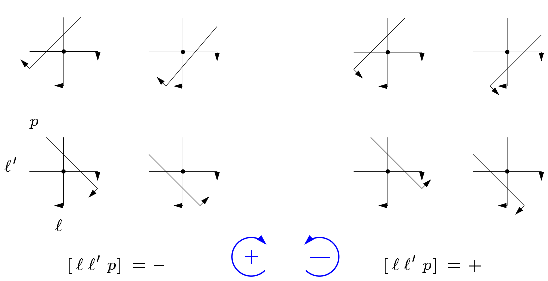

Figure 1: Pseudoline l added to the arrangement of lines 1 and 1'.

Thus the problem of extendability of uniform partial chirotopes of rank 3 is equivalent to the following problem:

Problem 1.2 *Given a partial, uniform chirotope of rank 3. Is there an arrangement $(p_1, \ldots, p_n)$ of oriented pseudolines such that $[ijk] = +(-)$ whenever the i, j, k bound the triangle as described above?*

Since there exists a polynomial-time test whether a chirotope satisfies the Grassmann-Plücker relations

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[1] From now on we will use the bracket notation as abbreviation for χ, i.e. $\chi(i_1, \ldots, i_r)$ becomes $[i_1 i_2 \ldots i_r]$.

this problem is in NP. Thus NP-hardness implies NP-completeness of Prob. 1.2 – which in turn implies NP-completeness of Prob. 1.1.

We will reduce Problem 1.2 to a variant of 3-SAT. Our construction is inspired by Richter-Gebert's proof ([7]) that the problem to decide whether a given matroid is orientable is NP-complete.

2 A variant of 3-SAT

Problem 2.1 *(NOT-ALTERNATING-3SAT) Given a set of ordered boolean variables $X = (x_1, \ldots, x_n)$ and a set S of m three-clauses. Is there a truth assignment for the elements of X such that in none of the clauses the three literals alternate, i.e. are $(false, true, false)$ and $(true, false, true)$?*

This problem is known to be NP-complete (cf. [7], [5]). Note that it is essential to have a total order on the indices of X, which induces an order on the literals of each clause. In our construction, each clause will correspond to a pair of pseudolines that have at least one crossing for each $false/true$ (or $true/false$) transition in a clause. Thus, alternating clauses would force this pair of pseudolines to cross twice, something which is forbidden by the definition.

The next couple of sections introduce the necessary constructions on more general terms. We will later use these building blocks to actually do our construction.

3 The frame of reference

The frame of reference into which we embed our construction will be a rectangular grid $\mathcal{G}_{n,m}$ contained in an oriented matroid $\mathcal{F}_{n,m}$ as visualized in Fig. 2: We shall have two ordered sets of lines, *vertical* and *horizontal* ones, and all lines belonging to the same set shall have the same orientation in the grid.

So, let $\mathcal{F}_{n,m}$ be the oriented matroid with elements $1, \ldots, n$, the so-called 'verticals', $1', \ldots, m'$, the so-called 'horizontals', and 0, the line at infinity. Impose the canonical order on these two sets: $1 < 2 < \cdots < n$, and $1' < 2' < \cdots < m'$, referring to their elements by using indices i, j, k, $i < j < k$ and i', j', k', $i' < j' < k'$, resp.[2]

4 Adding pseudolines

We will now enlarge this newly defined oriented matroid by placing additional pseudolines for which only partial information is given. Their location is, therefore, constrained, yet possesses a certain degree of freedom.

To simplify the description, we call the intersection point between a horizontal i and a vertical j' in our

[2]In this extended abstract we will omit the basis orientations necessary for our construction, unless it is crucial for the understanding of the proof.

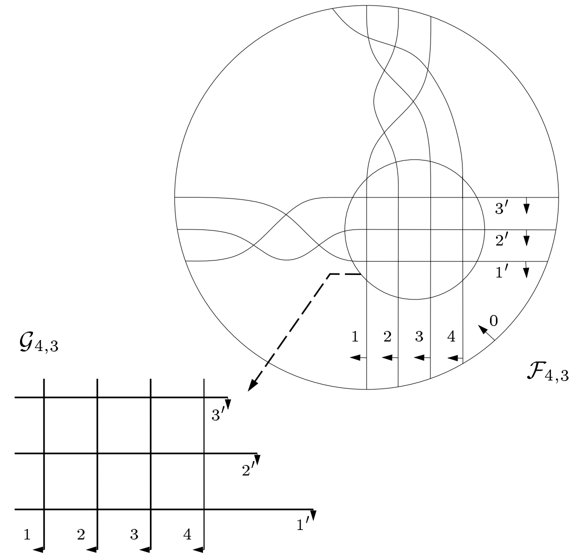

Figure 2: Instance of the 4×3 grid

rectangular grid $\mathcal{G}_{n,m}$ *vertex* $i \wedge i'$ of $\mathcal{G}_{n,m}$. We write $[i', j']_i$ short for the segment from $i \wedge i'$ to $i \wedge j'$ on i; $[i, j]_{i'}$ denotes the segment from $i \wedge i'$ to $j \wedge i'$ on i', see Fig. 3.

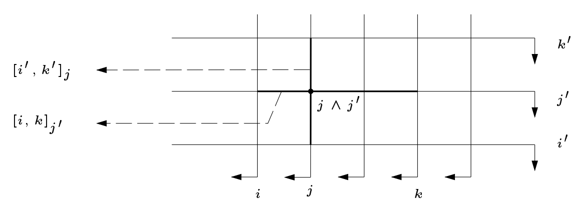

Figure 3: Vertices and segments in a rectangular grid.

Diagonal lines. Suppose we add just a single basis orientation for a new line p, $[i\, j'\, p] = +1$, say. In Fig. 1, we illustrated already the four options we have to draw this new pseudoline in our picture with respect to $i \wedge j'$.

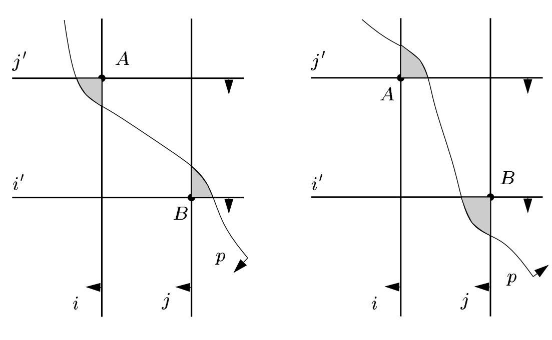

Figure 4: p is diagonal passing $i \wedge j'$ and $j \wedge i'$

Having two conditions for p, $[i\, j'\, p] = +1$, $[j\, i'\, p] = -1$, say, can reduce these possibilities to exactly two, see Fig. 4. Looking back at Fig. 1, it becomes clear that these two are indeed the only options. Note that p either cuts interior of the segments $[i,j]_{k'}$ or interior of the segments $[i',j']_k$ for all $i<k<j$ and $i'<k'<j'$.

This concept can be easily extended, see Fig. 5. We call a pseudoline p a *diagonal passing* $i \wedge k'$*,* $j \wedge j'$ *(and* $k \wedge i'$*)* whenever $[\,i\,k'\,p\,] = -[\,j\,j'\,p\,]\,(=[\,k\,i'\,p\,])$ holds for some pseudolines $i<j(<k),(i'<)j'<k'$.

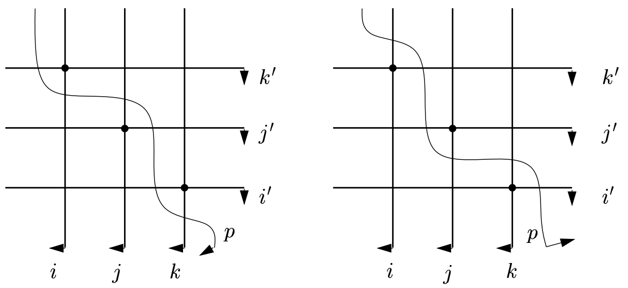

Figure 5: p is diagonal passing $i \wedge k'$, $j \wedge j'$ and $k \wedge i'$

Parallel lines. Now we include chirotope information for some additional pseudoline p such that it lies between two given pseudolines (either both vertical or both horizontal) inside our rectangular grid $\mathcal{G}_{n,m}$, cf. Fig. 6. The crucial property is that p intersects the interiors of the segments $[a',b']_i$ for all i.

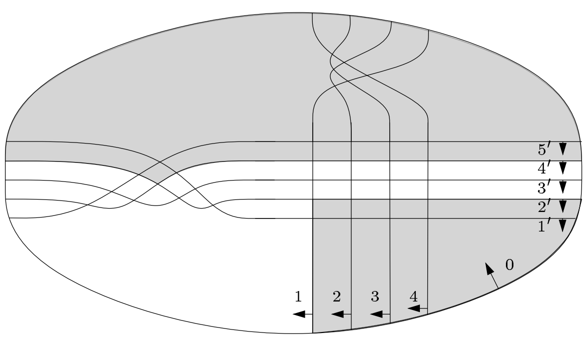

Figure 6: A horizontal parallel in $[a',b']$ cannot enter the shaded region.

Note that p has the same orientation as the other horizontal pseudolines. For the sake of brevity we will call such a pseudoline *horizontal parallel in* $[a',b']$. (It is completely open where p intersects the pseudolines $i', a'<i'<b'$. It will certainly intersect the pseudolines $i', i' \leq a'$ or $i' \geq b'$ *outside* $\mathcal{G}_{n,m}$.)

In a very similar fashion we can add partial chirotope information which will tell us about the existence of a *vertical parallel in* $[a,b]$.

Furthermore, it is feasible to include pseudolines with the same properties except that their orientation is pointing in the other direction. We call such a pseudoline *horizontal* (*vertical*) antiparallel in $[a',b']([a,b])$.

We must issue a warning: so far, we have not excluded the possibility that the added pseudoline will be *identical* to one of the already existing grid lines. We will cross this bridge later.

5 The construction

We are now ready to encode an instance of Problem 2.1 into an extendability problem. For this we need a switch which sets each variable true or false, and a structure to encode the actual clauses. In a final step, we will get a realization (and hence a full chirotope) if and only if the Problem 2.1 has an admissible assignment of boolean variables.

Let $X=(x_1,\ldots,x_n)$ be the sequence of boolean variables, and let $C_1,\ldots,C_m$ be the set of clauses. The frame $\mathcal{F} := \mathcal{F}_{3m+3,3n}$ contains the rectangular grid $\mathcal{G}_{3m+3,3n}$. This is large enough for our construction: for each variable x_i from X, we reserve three consecutive horizontal lines (rows) a_i,b_i,c_i of $\mathcal{G}$. For each clause C_j, we reserve three vertical lines (columns) $1_j,2_j,3_j$ of $\mathcal{G}$. In addition, we reserve three vertical lines $1,2,3$ for encoding the switches that resemble the boolean variables. Figure 7 sketches the global situation.

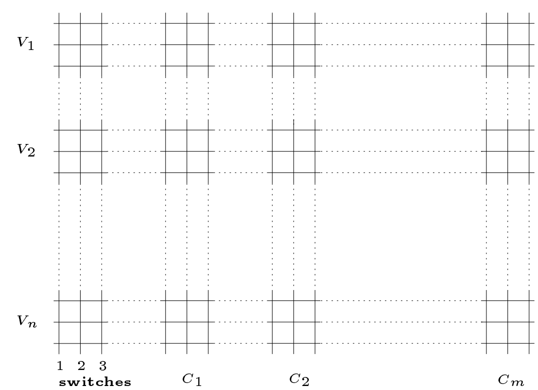

Figure 7: The structure to embed the NON-ALTERNATING-3SAT instance

The switch. The idea is to give partial information such that exactly two well-defined configurations can occur. One will then represent the case 'x_i is true', the other 'x_i is false'.

For every variable x_i we do the following:

First, we consider the rows a_i,b_i,c_i and add elements X_i and X_i' as *horizontal parallels in* $[a_i,c_i]$, representing x_i and $\neg x_i$. This alone does not give us any control over the location of the intersections points of X_i, X_i', b_i yet.

Secondly, for this very reason, we introduce two new *diagonals* U_i and W_i passing $1 \wedge c_i$ and $2 \wedge a_i$, and $2 \wedge c_i$ and $3 \wedge a_i$, resp., together with the following information: $[\,U_i\,a_i\,W_i\,] = [\,U_i\,b_i\,W_i\,] = [\,U_i\,c_i\,W_i\,] = +$, which guarantees that they have opposite orientation and do not intersect inside our little gadget.

Finally, we force U_i and W_i to meet X'_i, b_i, X_i in the same order. Since U_i and W_i have opposite orientation, X'_i, b_i, X_i must pairwise intersect between U_i and W_i inside the 3×3-grid-gadget.

The two possible situations in our realization are shown in Fig. 8. We associate the situation on the right with $x_i = false$, and the situation on the left with $x_i = true$.

We make the following crucial observation:

Lemma 5.1 *If we are in the situation $x_i = true$ then for all $3 \leq k \leq 3m + 3$ the line X'_i cuts the interval $[a_i, b_i]_k$ and the line X_i cuts the interval $[b_i, c_i]_k$. For $x_i = false$ the situation is reversed.*

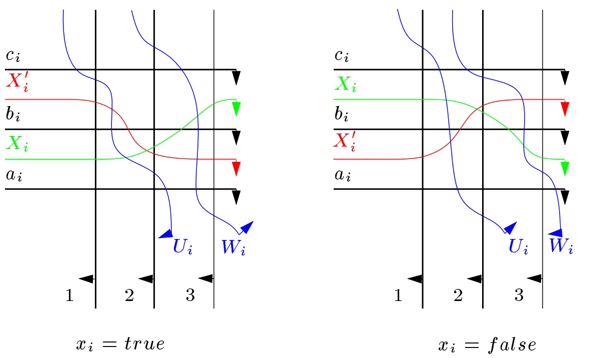

Figure 8: The two states of a switch.

The clauses. Lemma 5.1 implies that the actual literal l_i (i.e. x_i or $\neg x_i$) is represented by the line which cuts $[b_i, c_i]_k$, resp. $[a_i, b_i]_k$ for all $3 \leq k \leq 3m+3$ if and only if l_i is *true*, resp. *false*. From now on we refer to this line by L_i.

Remember that for each clause C_j we reserved three vertical lines $1_j, 2_j, 3_j$. The clause C_j consists of three literals. For each literal l_i (represented by L_i) that appears in the clause C_j we enlarge our chirotope with a diagonal T_i^j passing $1_j \wedge c_i$, $l \wedge b_i$ and $3_j \wedge a_i$, and satisfying $[T_i^j \, b_i \, l] = +$.

Now consider an arrangement by the pseudolines for T_i^j of our arrangement generated so far. Depending on the state of l we can only have one of the two situations shown in Fig. 9. The proof of the following lemma is again straightforward.

Lemma 5.2 *If we are in the situation $l = false$, then the line T_i^j cuts the interval $[1_j, 2_j]_{b_i}$. Otherwise T_i^j cuts the interval $[2_j, 3_j]_{b_i}$.*

Now we are almost done. There is only one missing element of our matroid for each clause. Assume that clause C_j consists of literals coming from variables x_{i_1}, x_{i_2} and x_{i_3}.

We add C_j, a *vertical parallel* in $[1_j, 3_j]$, satisfying

$$[C_j \, T_i^j \, b_{i_1}] = [C_j \, T_i^j \, b_{i_2}] = [C_j \, T_i^j \, b_{i_3}] = +.$$

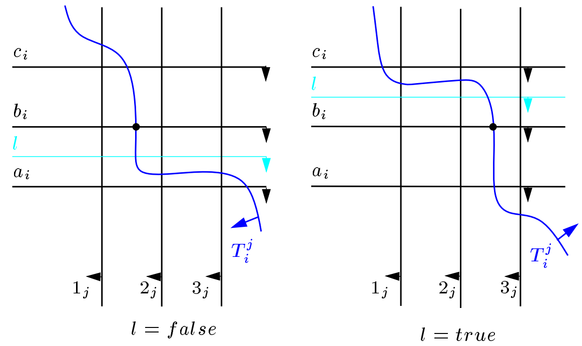

Figure 9: Connecting a switch and a clause

Extending our arrangement of pseudolines by the lines $C_1, \ldots, C_m$ is only possible if the corresponding literals in the clauses do not alternate. Alternating literals would force the line C_j to cross the line 2_j twice which is forbidden by the axioms. In all other cases the pseudolines are insertable.

This completes the proof of our result: from any pseudoline arrangement corresponding to our constructed matroid we can immediately read off an admissible assignment of truth values for the encoded instance of Problem 2.1. Conversely, every admissible assignment of truth values corresponds to a possible pseudoline arrangement that 'contains' the partial chirotope. Thus we have proved:

Theorem 5.1 *The problem of testing extendability of a partial chirotope is NP-complete.*

References

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