Turán's Theorem in sparse random graphs

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Abstract

We prove the analogue of Turán's Theorem in random graphs with edge probability $p(n) \gg n^{-1/(k-1.5)}$. With probability $1-o(1)$, one needs to delete approximately $\frac{1}{k-1}$–fraction of the edges in a random graph in order to destroy all cliques of size k.

1 Introduction

Turán's theorem [11], one of the starting points of extremal graph theory, asserts that for any given integer $k \geq 2$, one needs to delete at least $(\frac{1}{k-1} - O(\frac{1}{n}))\binom{n}{2}$ edges of the complete graph K_n in order to destroy all of its k-cliques. For an extension of Turán's Theorem to base graphs other than K_n, we introduce the following definition. Given a graph G, let $\mathrm{ext}(G,k)$ denote the minimum number of edges one needs to delete from G in order to obtain a K_k-free subgraph. (Note that we defined $\mathrm{ext}(G,k)$ instead of the more widely used *Turán number* $\mathrm{ex}(G,k)$; the connection is trivial through the identity $\mathrm{ex}(G,k) + \mathrm{ext}(G,k) = |E(G)|$.)

It seems very natural to investigate $\mathrm{ext}(G,k)$ when G is a typical, i.e., random graph on n vertices. In this paper, we will consider the Erdős-Rényi $G(n,p)$ model, which is perhaps the most well known model for random graphs. The random graph $G(n,p)$ on a vertex set of cardinality n is obtained by drawing an edge between any two vertices with probability p, independently. For $p = 1/2$, all graphs have equal probability and thus the distribution is uniform. In general, though, $p = p(n)$ can be a decreasing function of n. The study of $\mathrm{ext}(G(n,p),k)$ has recently become one of the more popular topics in the theory of random graphs; see the excellent monograph of Janson, Łuczak and Ruciǹski [6, Chapter 8] for a thorough discussion.

It is fairly easy to obtain an upper bound on $\mathrm{ext}(G(n,p),k)$ by following Turán's construction. Partition V into $k-1$ set $P_1, \ldots, P_{k-1}$ of sizes as equal as possible. Then delete all edges contained

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within each P_i which makes the remaining graph K_k-free — with still a lot of edges. More precisely, the number of edges deleted is a binomial random variable with expectation $\frac{1}{k-1}\binom{n}{2}p + O(n)$. By standard estimates on the tails of the binomial distribution, the probability that there are at most $(\frac{1}{k-1} + \epsilon)\binom{n}{2}p$ edges within the P_is is at least $1 - \exp(-\Theta(\epsilon^2 n^2 p))$ for any positive constant ϵ and any $p \gg \log n/n^2$ (see for example [6]).

It has been conjectured that for sufficiently large p, this upper bound is essentially tight. The following conjecture is the special case of Conjecture 8.11 from [6] and certainly seems plausible. It has the rough meaning that concerning Turán's Theorem $G(n,p)$ behaves like K_n when p is larger than an appropriately chosen threshold probability.

Conjecture 1.1 *For every integer $k \geq 3$ and for every $\eta > 0$, there exists a constant $K = K(\eta, k)$ such that*

$$\mathbf{P}\left(\operatorname{ext}(G(n,p),k) \leq \left(\frac{1}{k-1} - \eta\right)\binom{n}{2}p\right) \to 0, \tag{1}$$

provided $p = p(n) \geq Kn^{-\frac{2}{k+1}}$.

Here the exponent $\frac{-2}{k+1}$ is the negative reciprocal of the so-called *maximum K_2-density* of K_k. In particular, the threshold $n^{-\frac{2}{k+1}}$ for p in Conjecture 1.1 is the edge-probability around which more copies of K_k start to appear in $G(n,p)$, than edges. Certainly, for an edge-probability $p = cn^{-\frac{2}{k+1}}$ with a small enough constant c we can delete significantly less than $1/(k-1)$-portion of the edges of $G(n,p)$ and get rid of all copies of K_k by simply removing one edge from each K_k.

Conjecture 1.1 has been investigated by several researchers. It has been confirmed for $k = 3$ by Frankl and Rödl [3], and for $k = 4$ by Kohayakawa, Łuczak and Rödl [9]. The proof for K_4 is particularly technical and lasts over thirty pages. It also suggests a method to tackle the conjecture through variants of Szemerédi's Regularity Lemma and an even more general conjecture about certain ϵ-regular graphs.

For p constant, (1) is an easy consequence of Szemerédi's Regularity Lemma. The main achievement of this paper is the confirmation of (1) for all $p \geq Cn^{-1/(k-1.5)}$. Our argument is relatively short and does not use the Regularity Lemma. As a byproduct of our proof, we also establish the strong concentration of $\operatorname{ext}(G(n,p),k)$ around $\frac{1}{k-1}\binom{n}{2}p$.

Our main result is the following.

Theorem 1.2 *For every $\epsilon > 0$ and integer $k \geq 4$ there exists a constant $C = C(\epsilon, k)$, such that*

$$\mathbf{P}\left(|\operatorname{ext}(G(n,p),k) - \frac{1}{k-1}\binom{n}{2}p| \geq \epsilon\binom{n}{2}p\right) \leq 2\exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right),$$

provided $np^{k-1.5} \geq C$.

For a limited range of p, Theorem 1.2 justifies Conjecture 1.1 in a convincing way. Not only the right hand side is $o(1)$, but it is close to be the best bound one can get for any non-trivial event concerning $G(n,p)$. Indeed, notice that the probability that $G(n,p)$ is empty is $\exp(-\frac{1}{2}n^2p)$, so the best bound one can get for any non-trivial event is $\exp(-\Theta(n^2p))$.

Finally, let us mention that Kohayakawa, Rödl and Schacht [10], independently of our work, prove a more general theorem which implies the conjecture when $p > n^{-1/(k-1)}\log^c n$ for some constant c. This is a slightly weaker bound than the one in Theorem 1.2 and their method is very different from ours. The proof of Theorem 1.2, however, uses the nontrivial fact [9] that Conjecture 1.1 holds for $k=4$. As we would like to emphasize the simplicity of our approach, we point out that it is possible to avoid the use of [9] and obtain a relatively short, self-contained proof of (1) for $p \geq Cn^{-1/(k-1)}\log^{1/(k-1)} n$ (see Section 4).

In the paper we do not make an attempt to optimize the multiplicative constants, for example in Theorem 1.2 the constant $1/16$ in the exponent could easily be improved to $1/(4+\delta)$ with $\delta > 0$ arbitrary. We use the O, o, Ω and ω notations in the usual sense, i.e. for example $f = o(g)$ iff $f/g \to 0$ iff $g = \omega(f)$. Throughout the paper log denotes the natural logarithm. The rest of the paper is organized as follows. In the next section we present some of the main ideas of the proof with the key lemmas. The rest of the proof, i.e. the verification of these lemmas follows in Section 3. The last section contains several important remarks.

2 The main ideas

Set $Y_k = \mathrm{ext}(G(n,p),k)$ and let μ_k be the expectation of Y_k. As the upper tail probability in Theorem 1.2 is already discussed in the paragraphs preceding Conjecture 1.1, our task is to bound the lower tail probability, namely to show

$$\mathbf{P}\left(Y_k \leq \left(\frac{1}{k-1} - \epsilon\right)\binom{n}{2}p\right) \leq \exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right), \tag{2}$$

provided $np^{k-1.5} \geq C$.

A key ingredient of our proof is the strong concentration of the random variable Y_k around its mean. This fact is a straightforward consequence of a standard Azuma-Hoeffding type martingale lemma, first proved in this form by Kahn [7]. Here we cite the simpler formulation of Kim [8].

We say that a function $Y = Y(t_1, \ldots, t_m)$ has Lipschitz coefficient L if $|Y(t) - Y(t')| \leq L$ for any two vectors t and t' which differ at exactly one coordinate.

Lemma 2.1 *[8, Corollary 3.2] Let $t_1, \ldots, t_m$ be independent identically distributed (i.i.d.) binary random variables with mean p. For any function $Y = Y(t_1, \ldots, t_m)$ with Lipschitz coefficient 1,*

$$\mathbf{P}(Y \leq \mu - \lambda) \leq \exp\left(-\rho\lambda + \rho^2 pm\right),$$

where μ denotes the expectation of Y and $0 < \rho \le \log 2$.

The random variables t_i in our case are the $\binom{n}{2}$ i.i.d. characteristic random variables of the edges of $G(n,p)$. As a function in these variables, Y_k has Lipschitz coefficient 1. Applying Lemma 2.1 with $m = \binom{n}{2}$, $\rho = \frac{\lambda}{2mp}$ and $\lambda = \epsilon mp$ we obtain the following corollary.

Lemma 2.2 *For any integer $k \ge 2$ and $2\log 2 \ge \epsilon > 0$*

$$\mathbf{P}\left(Y_k \le \mu_k - \epsilon\binom{n}{2}p\right) < \exp\left(-\frac{\epsilon^2 p}{4}\binom{n}{2}\right).$$

We prove (2) by induction on k. We start the induction at $k = 4$. Conjecture 1.1 has been confirmed for this case [9] so we know that for any positive constant η there is a constant $K = K(\eta, 4)$ such that $\mathbf{P}(Y_4 \le (\frac{1}{3} - \eta)\binom{n}{2}p) \to 0$ provided $p \ge Kn^{-2/5}$. This implies that for every positive constant ϵ, there is a constant $C = C(\epsilon, 4)$, such that $\mu_4 \ge (\frac{1}{3} - \frac{\epsilon}{2})\binom{n}{2}p$ provided $np^{2.5} \ge C$. Then, by Lemma 2.2,

$$\mathbf{P}\left(Y_4 \le \left(\frac{1}{3} - \epsilon\right)\binom{n}{2}p\right) \le \mathbf{P}\left(Y_4 - \mu_4 \le -\frac{\epsilon}{2}\binom{n}{2}p\right) \le \exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right).$$

This completes the proof for the base case $k = 4$.

Assume now that for some $k \ge 4$ there exists a constant $C(\epsilon, k)$ for any $\epsilon > 0$ such that (2) holds provided $np^{k-\frac{3}{2}} \ge C(\epsilon, k)$. Fix an $\epsilon > 0$ and suppose that $np^{k-\frac{1}{2}} > C$, where $C = C(\epsilon, k+1)$ is a large constant to be specified during the proof. In order to prove the theorem for $(k+1)$, notice that by the same large deviation argument used in the previous paragraph, it suffices to show that

$$\mu_{k+1} \ge \left(\frac{1}{k} - \frac{\epsilon}{2}\right)\binom{n}{2}p. \tag{3}$$

Indeed, if (3) holds, then

$$\mathbf{P}\left(Y_{k+1} \le \left(\frac{1}{k} - \epsilon\right)\binom{n}{2}p\right) \le \mathbf{P}\left(Y_{k+1} - \mu_{k+1} \le -\frac{\epsilon}{2}\binom{n}{2}p\right) \le \exp\left(-\frac{\epsilon^2 p}{16}\binom{n}{2}\right),$$

where the last inequality again follows from Lemma 2.2.

The following definition plays a crucial role in our proof. For an integer $k \ge 2$ and positive $\delta, p \le 1$, we say that a graph G on n vertices is (k, δ, p)-*normal* if the following three conditions hold.

(i) For every set X of size $x \ge \delta np$, X contains at least $(1-\delta)p\binom{x}{2}$ edges.

(ii) For any pair of vertices u and v, $\mathrm{codeg}(u,v) \le (1+\delta)np^2$, where $\mathrm{codeg}(u,v)$ is the number of common neighbors of u and v.

(iii) For every set X of size $x \ge \delta np$, one needs to delete at least $(\frac{1}{k-1} - \delta)\binom{x}{2}p$ edges from the subgraph G_X induced by X in order to destroy all k-cliques in this subgraph.

If there is no danger of ambiguity we just call a (k,δ,p)-normal graph *normal.*

Remark. The first two conditions are rather standard properties, frequently seen in the theory of random graphs. The power of the definition lies in the third condition, which lays ground for an inductive argument. Observe also that for $k=2$ the third condition is redundant. This fact is important in case we want to obtain a self-contained proof by starting the induction at $k=3$ rather than $k=4$. (See the first remark in Section 4.)

Inequality (3) is a straightforward consequence of the following two lemmas.

Lemma 2.3 *For every $\delta>0$ and integer $k\geq 2$ there is a constant $C'=C'(\delta,k+1)$, such that if $\min\{np^{k-\frac{1}{2}},np^2/\log n\}\geq C'$ then $\mathbf{P}(G(n,p)$ is (k,δ,p)-normal$)>1-\delta$.*

Lemma 2.4 *If G is a (k,δ,p)-normal graph on n vertices and $pn\geq 4/\delta$, then $\mathrm{ext}(G,K_{k+1})\geq (\frac{1}{k}-11\delta)\binom{n}{2}p$.*

Indeed, if Lemmas 2.3 and 2.4 hold, then by setting $\delta=\epsilon/24$, we have

$$\mu_{k+1}\geq(1-\delta)\left(\frac{1}{k}-11\delta\right)\binom{n}{2}p\geq\left(\frac{1}{k}-\frac{\epsilon}{2}\right)\binom{n}{2}p, \tag{4}$$

proving (3).

Notice that the conditions $np^2=\omega(\log n)$ and $np=\omega(1)$ in the lemmas are trivially satisfied in the current situation as we basically assumed $np^{k-\frac{1}{2}}=\omega(1)$ for some $k\geq 4$ — so to some extent, these conditions are redundant. Still, we prefer to present these lemmas in their best possible form, as they might be of independent interest.

3 Proof of the lemmas

3.1 Proof of Lemma 2.3

We apply the following estimate a couple of times.

$$\begin{aligned}\sum_{x\geq\delta np}^{n}\binom{n}{x}\exp\left(-\frac{\delta^2p}{16}\binom{x}{2}\right) &< \sum_{x\geq\delta np}^{n}\exp(x(\log n-\delta^2px/64))<\sum_{x\geq\delta np}^{n}\exp(x(\log n-\delta^3p^2n/64))\\ &< \sum_{x\geq\delta np}^{n}\exp(x\log n\,(1-\delta^3C'/64))<\sum_{x\geq\delta np}^{n}\left(\frac{\delta}{3n}\right)^x<\frac{\delta}{3},\end{aligned}\tag{5}$$

if $C'=C'(\delta,k+1)$ is chosen large enough.

(i) Let X be a subset of the vertices with $|X|=x\geq\delta np$ and denote by D_X the random variable of the number of edges in X. Then D_X is a binomial random variable $B(\binom{x}{2},p)$ with expectation $\binom{x}{2}p$. By standard estimates on the lower tail of the binomial distribution, we have $\mathbf{P}(D_X < (1-\delta)\binom{x}{2}p) < \exp(-\frac{\delta^2 p}{2}\binom{x}{2})$. Thus the probability that there exists a set X contradicting the first condition of normality is at most $\sum_{x\geq\delta np}^{n}\binom{n}{x}\exp(-\frac{\delta^2 p}{2}\binom{x}{2}) < \delta/3$ by (5).

(ii) Let $u, v \in V$. Then $\text{codeg}(u,v)$ is a binomial random variable $B(n-2, p^2)$ with mean $(n-2)p^2$. Thus the standard upper tail estimate gives $\mathbf{P}(\text{codeg}(u,v) > (1+\delta)np^2) < \exp(-c\delta^2 np^2)$, where $c = c(\delta)$ is a constant. So the probability that *there is* a pair u, v with $\text{codeg}(u,v) > (1+\delta)np^2$ is at most $n^2\exp(-c\delta^2 np^2) < \exp(2\log n - c\delta^2 C' \log n) < \delta/3$, for large enough C'.

(iii) Finally, let X be a subset of the vertex set of size $x \geq \delta np$. We would like to have that $xp^{k-\frac{3}{2}} \geq (\delta np)\cdot p^{k-\frac{3}{2}} > C(\delta, k)$, so choose $C' = C'(\delta, k+1) > C(\delta,k)/\delta$. Here $C(\delta,k)$, the constant from Theorem 1.2, exists by our induction hypothesis. Then in the random graph $G|_X = G(x,p)$ with probability at least $1-\exp(-\frac{\delta^2 p}{16}\binom{x}{2})$ one needs to delete at least $(\frac{1}{k-1}-\delta)\binom{x}{2}p$ edges in order to destroy all K_k in G_X. Thus the probability that there exists a subset X violating the condition is at most $\sum_{x\geq\delta np}^{n}\binom{n}{x}\exp(-\frac{\delta^2 p}{16}\binom{x}{2}) < \delta/3$ by (5).

Hence if $C' = C'(\delta, k+1)$ is chosen to be large enough, all three conditions hold simultaneously with probability more than $1-\delta$. □

3.2 Proof of Lemma 2.4.

Consider a (k,δ,p)-normal graph $G = G(V,E)$ and let E_R be a set of edges such that $E_B = E\backslash E_R$ does not contain a K_{k+1} and $|E_R| = \text{ext}(G, K_{k+1})$, i.e. the cardinality of E_R is minimum. We will call the edges in E_R red and the edges in E_B blue. For an $X \subset V$, G_X denotes the subgraph induced by X.

Let N_1, N_2, and N_3 denote the number of triangles in G with exactly one, two and three red edges, respectively. Our claim will follow from double counting arguments which estimate N_1 from both sides.

For each vertex v, let R_v (resp. B_v) be the set of neighbors of v which are connected to v by a red (resp. blue) edge. Furthermore, let f_v be the number of red edges in the graph induced by B_v. It is clear that

$$N_1 = \sum_{v\in V} f_v. \tag{6}$$

As E_B does not contain an K_{k+1}, the deletion of the red edges destroys all K_k's in the subgraph induced by B_v. In other words, $f_v \geq \text{ext}(G_{B_v}, K_k)$. We consider two cases. First, if $|B_v| \geq \delta np$, then by condition *(iii)* in the definition of a normal graph

$$f_v \geq \text{ext}(G_{B_v}, k) \geq \left(\frac{1}{k-1}-\delta\right)\binom{|B_v|}{2}p. \tag{7}$$

Second, if $|B_v| < \delta np$, then trivially

$$\left(\frac{1}{k-1}-\delta\right)\binom{|B_v|}{2}p \leq \delta^2 n^2 p^3. \tag{8}$$

From (6), (7) and (8), we obtain

$$N_1 = \sum_v f_v \geq \sum_{v \in V} \left(\frac{1}{k-1} - \delta \right) \binom{|B_v|}{2} p - \delta^2 n^3 p^3. \tag{9}$$

In order to provide an upper bound for N_1, for each red edge $e = \{a, b\}$, we count the number of co-neighbors of a and b in the graph (V, E_B), spanned by the blue edges. To start, notice that

$$N_1 = \sum_{\{a,b\} \in E_R} |B_a \cap B_b| = \sum_{e \in E_R} (\mathrm{codeg}(e) - m(e)) \leq |E_R|(1+\delta)np^2 - \sum_{e \in E_R} m(e),$$

where $m(a,b) = |R_a \cap B_b| + |B_a \cap R_b| + |R_a \cap R_b|$ and the last inequality follows from the co-degree property of normal graphs.

Now we are going to find a lower bound for $\sum_{e \in E_R} m(e)$. First, note that

$$\sum_{e \in E_R} m(e) = 2N_2 + 3N_3.$$

On the other hand,

$$2N_2 + 3N_3 \geq N_2 + 3N_3 = \sum_{v \in V} e(R_v) \geq (1-\delta) \sum_{v \in V} \binom{|R_v|}{2} p - \delta^2 n^3 p^3,$$

where $e(R_v)$ denotes the number of edges (of both colors) in R_v. Here the last inequality follows from the first property of normal graphs and the trivial observation that $(1-\delta)\binom{|R_v|}{2}p \leq \delta^2 n^2 p^3$ if $|R_v| < \delta np$. Thus

$$N_1 \leq (1+\delta)np^2|E_R| - (1-\delta) \sum_{v \in V} \binom{|R_v|}{2} p + \delta^2 n^3 p^3. \tag{10}$$

Combining (9) and (10) yields

$$\sum_{v \in V} \left(\frac{1}{k-1} - \delta \right) \binom{|B_v|}{2} p - \delta^2 n^3 p^3 \leq (1+\delta)np^2|E_R| - (1-\delta) \sum_{v \in V} \binom{|R_v|}{2} p + \delta^2 n^3 p^3. \tag{11}$$

It is now a fairly routine calculation to complete the proof. Let r (resp. b) be the average red (resp. blue) degree of the vertices. Then $|E_R| = nr/2$. By the Cauchy-Schwarz inequality, it follows from (11) that

$$\left(\frac{1}{k-1} - \delta \right) n \binom{b}{2} p - \delta^2 n^3 p^3 \leq (1+\delta) n^2 p^2 r/2 - (1-\delta) n \binom{r}{2} p + \delta^2 n^3 p^3.$$

It is clear that $r + b = d$, the average degree 63 of G. The first property of normal graphs then yields that $b = d - r \geq (1-\delta)(n-1)p - r$, so by dividing both sides by $np/2$, we obtain that

$$\left(\frac{1}{k-1} - \delta \right) ((1-\delta)(n-1)p - r)((1-\delta)(n-1)p - r - 1) \leq (1+\delta)npr - (1-\delta)r(r-1) + 4\delta^2 n^2 p^2. \tag{12}$$

We can assume that $r \leq (1-1.5\delta)np$ and $\delta \leq 1$ otherwise there is nothing to prove. Then $np \geq 4/\delta$ implies $(\frac{1}{k-1}-\delta)((1-\delta)(n-1)p-r)((1-\delta)(n-1)p-r-1) \geq \frac{1}{k-1}(np-r)^2 - 4\delta n^2p^2$. Similarly $(1+\delta)npr - (1-\delta)r(r-1) \leq npr - r^2 + 3\delta n^2p^2$. Finally, $4\delta^2n^2p^2 \leq 4\delta n^2p^2$. Combining these with (12) we infer

$$\frac{1}{k-1}(np-r)^2 - 4\delta n^2p^2 \leq npr - r^2 + 7\delta n^2p^2,$$

which is equivalent to

$$\left(\frac{1}{k-1}+1\right)r^2 - \left(\frac{2}{k-1}+1\right)(np)r + \left(\frac{1}{k-1}-11\delta\right)(np)^2 \leq 0.$$

Solving the quadratic inequality for r we obtain that

$$\begin{aligned} r &\geq np\frac{\frac{k+1}{k-1} - \sqrt{(\frac{k+1}{k-1})^2 - \frac{4k}{k-1}(\frac{1}{k-1}-11\delta)}}{\frac{2k}{k-1}} \\ &= np\frac{\frac{k+1}{k-1} - \sqrt{1+\frac{4k}{k-1}11\delta}}{\frac{2k}{k-1}} \geq np\frac{\frac{k+1}{k-1} - (1+\frac{22k}{k-1}\delta)}{\frac{2k}{k-1}} \\ &= (\frac{1}{k} - 11\delta)np, \end{aligned}$$

This completes the proof of Lemma 2.4 and thus the proof of Theorem 1.2. □

4 Remarks

Base of the induction. In the proof, the base of our induction is $k=4$ and we need the (rather non-trivial) result [9] of Kohayakawa, Łuczak and Rödl in order to verify our hypothesis for this case. Should a self-contained proof be desirable, we could start the induction at $k=3$ and obtain the somewhat weaker bound $p = \omega(n^{-1/(k-1)}\log^{1/(k-1)} n)$ for the general case. This bound is essentially the same as the recent bound of Kohayakawa, Rödl and Schacht [10] proved by a different method. The logarithmic factor could actually be erased by some extra work.

To derive the independent proof we could start by observing that for $k=2$ the third condition of normality is redundant, thus the proof of Lemma 2.3 implies that $G(n,p)$ is $(2,\delta,p)$-normal with probability $1-\delta$ provided $np^2/\log n > C'$. Combining this with Lemmas 2.4 and 2.2 yields Conjecture 1.1 for $k=3$ in a slightly more restricted range, i.e. when $np^2/\log n > C$. The proof of the base of the induction is thus complete, while the general step follows from Lemmas 2.3, 2.4 and 2.2 the same way as in the paper.

Generalization and Refinement. It seems that with some extra work, our result can be extended in more than one way. First of all, we can replace k-cliques by other fixed graphs. The general problem is to determine how many edges should one delete from $G(n,p)$ to destroy all copies of a fixed graph H. Our inductive method carries over to prove an asymptotic version of the Erdős-Stone-Simonovits Theorem [2, 1] in sparse random graphs for several infinite families of H. More precisely, suppose that for every $\eta > 0$, $\mathbf{P}(ext(G(n,p),H) < (1/(\chi(H)-1)-\eta)\binom{n}{2}p) \to 0$ provided $p \gg n^{-1/\alpha}$. Let H' be a supergraph of H, such that $V(H') = V(H) \cup \{u\}$, $E(H'|_{V(H)}) = E(H)$ and $\chi(H') = \chi(H)+1$. Then an asymptotic Erdős-Stone Theorem is true for H' as well: for every $\eta > 0$, $\mathbf{P}(ext(G(n,p),H') < (1/(\chi(H')-1)-\eta)\binom{n}{2}p) \to 0$ provided $p \gg n^{-1/(\alpha+1)}$. Thus, for example, the existing results for cyles [4, 5] imply a sparse Erdős-Stone-Simonovits Theorem for the join $C_n \vee K_m$ of an arbitrary cycle and a complete graph, or for the "little house"-graph on five vertices (C_4 and a fifth vertex connected to two adjacent vertices of the C_4). Another direction of extension is to find the best bound on the error term η in Conjecture 1.1. In this conjecture and also in Theorem 1.2, the error terms are arbitrarily small constants. We believe that the best error term should be a constant negative power of n and determining this error term may be of independent interest. Our arguments can be refined to obtain results in this direction. Yet another line of extension is to study the case when k tends to infinity with n. Even for $p = 1/2$ it is not clear how large we can set k in Conjecture 1.1. The approach using Szemerédi's regularity lemma seems to be ineffective when $k \geq \log_* n$. On the other hand, from above, it can be shown that the conjecture is no longer true if $k \geq K \log n$ for some constant K. It seems plausible that our method could work for k up to $c \log n$ for some small constant c. Details about these extensions (some of which are rather involved and require additional ideas) will appear in a future paper.

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References

[1] P. Erdős, M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar* **1**, (1966) 51-57.

[2] P. Erdős, A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.*, **52**, (1946) 1087-1091.

[3] P. Frankl, V. Rödl, Large triangle-free subgraphs in graphs without K_4, *Graphs and Combinatorics*, **2** (1986), 135-144.

[4] P. E. Haxell, Y. Kohayakawa, T. Łuczak, Turán's extremal problem in random graphs: forbidding even cycles, *Journal of Combinatorial Theory, Ser. B* **64** (1995), 273-287.

[5] P. E. Haxell, Y. Kohayakawa, T. Łuczak, Turán's extremal problem in random graphs: forbidding odd cycles, *Combinatorica* **16** (1996), 133-163.

[6] S. Janson, T. Łuczak, A. Ruciǹski, Random Graphs, John Wiley and Sons, (2000).

[7] J. Kahn, Asymptotically good list-colorings, *J. Combin. Theory Ser. A* **73** (1996), no. 1, 1-59.

[8] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures & Algorithms* **7** (1995), 173-207.

[9] Y. Kohayakawa, T. Łuczak, V. Rödl, On K^4-free subgraphs of random graphs, *Combinatorica* **17** (1997), 173-213.

[10] Y. Kohayakawa, V. Rödl, M. Schacht The Turán theorem for random graphs, *submitted* (2002).

[11] P. Turán, Egy gráfelméleti szélsőértékfeladatról (in Hungarian), *Mat. Fiz. Lapok* **48**, (1941) 436-452.