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Point-Line Incidences in Space[*][†]

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Abstract

Given a set L of n lines in $\mathbb{R}^3$, joints are points in $\mathbb{R}^3$ that are incident to at least three *non-coplanar* lines in L. We show that there are at most $O(n^{5/3})$ incidences between L and the set of its joints.

This result leads to related questions about incidences between L and a set P of m points in $\mathbb{R}^3$: First, we associate with every point $p \in P$ the minimum number of planes it takes to cover all lines incident to p. Then the sum of these numbers is at most

$$O(m^{4/7}n^{5/7} + m + n) \ .$$

Second, if each line forms a fixed given non-zero angle with the xy-plane—we say the lines are *equally inclined*—then the number of (real) incidences is at most

$$O(\min\{m^{3/4}n^{1/2}\kappa(m), m^{4/7}n^{5/7}\} + m + n) \ ,$$

where $\kappa(m) = (\log m)^{O(\alpha^2(m))}$, and $\alpha(m)$ is the slowly growing inverse Ackermann function. These bounds are smaller than the tight

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Szemerédi-Trotter bound for point-line incidences in $\mathbb{R}^2$, unless both bounds are linear. They are the first results of that type on incidences between points and 1-dimensional objects in $\mathbb{R}^3$. This research was stimulated by a question raised by G. Elekes.

1 Introduction

What is the maximum number of incidences between m points and n lines in $\mathbb{R}^3$? We can answer that question by projecting the lines and points to some generic plane. In the projection, incidences in space are preserved, and no new ones are created. In this plane, we can apply the worst-case tight bound of $O(m^{2/3}n^{2/3} + m + n)$ for point-line incidences in $\mathbb{R}^2$—a classical result of Szemerédi and Trotter [16] (see also [5, 15]). Hence, the same bound holds for point-line incidences in $\mathbb{R}^3$. Moreover, since we can choose all points and lines in a common plane in $\mathbb{R}^3$ to begin with, the tight lower bound construction in the plane can be adopted right away.

Problem Set-up. Here we ask the question of what happens when the lines and points are 'truly in space.' In particular, we want to give little (or no) weight to incidences between a point and a set of lines that lie all in a common plane. To make this more concrete, let L be a set of lines in $\mathbb{R}^3$. For $p \in \mathbb{R}^3$, we denote by L_p the set of lines incident to p, i.e., $L_p := \{\ell \in L | \ell \ni p\}$.

We call p a *joint of* L, if L_p contains at least three non-coplanar lines (i.e., not all lines in L_p are contained in a single plane), and we let J_L denote the set of all joints of L. Joints in line arrangements have been investigated in [12], where a bound of $O(|L|^{1.643})$ for the number of joints was shown, thereby improving on the previous bound of $O(|L|^{1.75})$ in [4]. An easy grid-based construction shows that the number of joints in a set of n lines can be $\Omega(n^{3/2})$ [4, 11].

We let $c_p = c_p(L)$, the *plane-cover number*[1] *of* p, denote the minimum number of planes that contain all lines in L_p. Note that p is a joint iff $c_p \geq 2$.

For $P \subseteq \mathbb{R}^3$ a finite set of points, we set $I(P, L) := \sum_{p \in P} |L_p|$ and $I_c(P, L) := \sum_{p \in P} c_p(L)$. For $n := |L|$ and $m := |P|$, our initial discussion

[1] As a side remark, for given L, p, and $k \in \mathbb{N}$, deciding whether $c_p(L) \geq k$ is NP-complete: It is (linear time) equivalent to deciding whether k lines can cover a given planar set of n points, which has been shown to be NP-complete in [17].

entails a worst-case tight bound of $I(P,L) = O(m^{2/3}n^{2/3} + m + n)$.

Upper Bounds. Our main result is

$$I(J_L, L) = O(n^{5/3}) \tag{1}$$

(see Theorem 3.1). We use this bound to derive

$$I_c(P, L) = O(m^{4/7}n^{5/7} + m + n) \tag{2}$$

(see Theorem 4.1). Note that $c_p = \lceil |L_p|/2 \rceil$ if no three lines in L_p are coplanar. Hence, if no three lines in L through a common point are coplanar, then

$$I_c(P, L) - \frac{1}{2}|P| \leq \tfrac{1}{2}\, I(P, L) \leq I_c(P, L) \tag{3}$$

and, therefore, the bound in (2) is actually a bound for $I(P, L)$ under this precondition. In particular, no three lines in any L_p are coplanar if all lines are *equally inclined*, that is, each line in L forms a fixed given non-zero angle with the xy-plane. In that specific situation we provide the further improvement

$$\begin{gathered} I(P, L) \leq 2I_c(P, L) = \\ O(\min\{m^{3/4}n^{1/2}\kappa(m), m^{4/7}n^{5/7}\} + m + n)\ , \end{gathered} \tag{4}$$

where $\kappa(m) = (\log m)^{O(\alpha^2(m))}$, and $\alpha(m)$ is the slowly growing inverse Ackermann function (see Theorem 5.1). This latter question about equally inclined lines was raised by G. Elekes, and we are grateful to him for stimulating this research.[2]

For $n = o(m^2)$, the bounds in (2) and (4) are smaller than the tight Szemerédi-Trotter bound for point-line incidences in $\mathbb{R}^2$ (for $n = \Omega(m^2)$, all bounds are $O(n)$). These are the first results of that type on incidences between points and 1-dimensional objects in $\mathbb{R}^3$.

[2]Elekes was actually interested in a similar problem formulated in the complex space, whose interpretation in the reals involves incidences between points and *helices* of a certain kind in three dimensions. The case of equally inclined lines is the simpler real version of Elekes' problem.

Lower Bounds. We show that, for $m, n \in \mathbb{N}$ with $m = \Omega(n^{3/4})$, there are sets P of m points and L of n equally inclined lines in $\mathbb{R}^3$, such that

$$I(P, L) = \Omega(m^{2/3}n^{1/2}) \tag{5}$$

(see Theorem 7.1). This bound is superlinear in $(n + m)$ for m satisfying $m = \omega(n^{3/4})$ and $m = o(n^{3/2})$, and, along (3), it is also a lower bound for I_c.

Finally, we consider a 'competing' way of defining non-coplanar incidences, for sets L of arbitrary lines. For $p \in \mathbb{R}^3$, let $s_p = s_p(L)$ be the number of distinct planes that contain at least two lines in L_p, and let $\nu_p = \nu_p(L) := \sqrt{s_p}$. Define $I_\nu(P, L) := \sum_{p\in P} \nu_p(L)$. We show that $c_p = O(\nu_p)$, and we derive a worst case lower bound of

$$I_\nu(P, L) = \Omega(m^{1/2}n^{3/4}) \ . \tag{6}$$

This lower bound, however, does not apply to I_c, for which we have no lower bounds other than the one implied by (5).

Tools. The analysis exploits the Szemerédi-Trotter bound [16], the structure of reguli in 3-space (see, e.g., [14]), results from extremal graph theory for forbidden complete bipartite subgraphs (see, e.g., [10]), partition schemes from computational geometry (see, e.g., [5, 13]), and methods reminiscent of those developed in [4, 12] for the analysis of joints in line arrangements in space.

The derivation of the second bound for equally inclined lines uses a totally different approach. It transforms the sets P and L to a planar configuration involving points and circles, where circles are tangent to each other at the given points, and where the goal is to bound the number of such tangencies. This is handled using recent tools developed for arrangements of circles in [1, 9].

2 Prerequisites

We recall some of the tools we need for our proofs. On the way, we show that many lines in a common plane or in a common regulus are counter-productive to having many incidences between the lines and their joints.

Szemerédi-Trotter Bound [16]. Given a set L of n lines and a set P of m points, both in a common (2-dimensional) plane, we have[3]

$$I(P, L) = O(n^{2/3}m^{2/3} + n + m) \ . \tag{7}$$

Now let L be an arbitrary set of n lines in $\mathbb{R}^3$, and L_π be the subset of *all* lines from L that lie in some given plane π. Set $n_\pi := |L_\pi|$ and $m_\pi := |J_L \cap \pi|$. Note that $m_\pi \leq |L \setminus L_\pi| \leq n$, since every point $p \in J_L \cap \pi$ needs a line ℓ with $p \in \ell \in L \setminus L_\pi$, to ensure that p is indeed a joint, and ℓ cannot serve this purpose for any other point in π. Hence, the number of incidences between $J_L \cap \pi$ and L is at most

$$\begin{aligned} & |L \setminus L_\pi| + I(J_L \cap \pi, L_\pi) \\ \leq \ & n + O(m_\pi^{2/3} n_\pi^{2/3} + m_\pi + n_\pi) \\ = \ & O(n^{2/3} n_\pi^{2/3} + n) \ , \end{aligned}$$

and so

$$I(J_L, L) \leq I(J_{L\setminus L_\pi}, L \setminus L_\pi) + O(n^{2/3}n_\pi^{2/3} + n),$$

since all joints of L outside π remain joints in $L \setminus L_\pi$.

Reguli (see [14]). Two lines in $\mathbb{R}^3$ that are disjoint and not parallel are called *skew*. Given three pairwise skew lines ℓ_1, ℓ_2, ℓ_3, the set, $\sigma = \sigma(\ell_1, \ell_2, \ell_3)$, of lines intersecting all three lines is called a *regulus*. All lines in σ are pairwise skew. If $\ell_1', \ell_2', \ell_3'$ are in σ, then $\sigma^\perp = \sigma(\ell_1', \ell_2', \ell_3')$ constitutes another regulus, that is independent of the choice of the three lines in σ. (Note that the three generating lines, ℓ_1, ℓ_2, ℓ_3, of σ do *not* belong to σ, but rather to $\sigma^\perp$.)

$\bigcup_{\ell\in\sigma} \ell = \bigcup_{\ell\in\sigma^\perp} \ell$ is a *ruled surface* (which is a quadric) in $\mathbb{R}^3$, denoted by $\sigma^* = \sigma^*(\ell_1, \ell_2, \ell_3)$; σ and $\sigma^\perp$ are called the *generating families of* σ^* and we say that $\sigma^\perp$ is the *complementary regulus of* σ, and vice versa: $(\sigma^\perp)^\perp = \sigma$. Every point in σ^* is contained in exactly one line from σ and in exactly one line from $\sigma^\perp$. For any line ℓ in $\mathbb{R}^3$, either $\ell \in \sigma \cup \sigma^\perp$ (i.e., $\ell \subseteq \sigma^*$), or ℓ intersects σ^* in at most two points.

It follows that if $P \subseteq \sigma^*$ and L is any set of n lines, then

$$I(P, L) \leq 2(|P| + n),$$

[3] This bound is worst case tight: Choose P as the set of vertices of an $M \times M$ grid of points in the plane, for $M = \lfloor\sqrt{m}\rfloor$, and L as a set of n distinct lines that maximize the number of incidences with P; see [6].

by the following argument: Any point-line incidence (p, ℓ) is associated with p, if $\ell \in \sigma \cup \sigma^\perp$, and it is associated with ℓ, otherwise. In this way every point and every line is associated with at most two incidences.

If, moreover, all points in P are joints of L then

$$I(P, L) \leq 6n, \tag{8}$$

since then every point in P must be incident to at least one line not in $\sigma \cup \sigma^\perp$, and each such line is incident to at most two points in P, implying that $|P| \leq 2n$. It follows that if L_σ is the set of all lines from L in a regulus σ or its complementary regulus $\sigma^\perp$, then

$$I(J_L, L) \leq I(J_{L \setminus L_\sigma}, L \setminus L_\sigma) + 6n \ .$$

Forbidden Subgraphs. If a simple finite bipartite graph $G = (U \dot\cup V, E)$, $E \subseteq U \times V$, contains no (isomorphic copy of the) complete bipartite graph $K_{3,2}$ with the independent part of size 3 in U, then (see, e.g., [10, Theorem 9.5])

$$\begin{aligned} |E| &= O(|U| \cdot |V|^{2/3} + |V|) \\ |E| &= O(|V| \cdot |U|^{1/2} + |U|) \ . \end{aligned} \tag{9}$$

Let us see a simple application of that result from extremal graph theory. Let L be a set of n lines in $\mathbb{R}^3$, and let R be a family of reguli.[4] Then the number of pairs $(\ell, \sigma) \in L \times R$ with $\ell \in \sigma$—the number of incidences between L and R, if you like—is bounded by $O(n \cdot |R|^{2/3} + |R|)$, since three lines in L are contained in at most one common regulus $\sigma \in R$.

3 Joint-Line Incidences

This section is dedicated to the proof of the main theorem, with a discussion of some immediate implications.

Theorem 3.1 *For any set L of n lines in $\mathbb{R}^3$, we have $I(J_L, L) = O(n^{5/3})$.*

[4] Here it is important that a regulus consists only of one of the two generating families of lines of a ruled surface!

Let L be a set of n lines in $\mathbb{R}^3$. For convenience, *we assume that no two lines in L are parallel.* We may do so since this property can be attained by an appropriate projective transformation, without affecting incidences and coplanarities.

By the discussion in the previous section, removing a set L' of all k lines in a given plane or regulus will decrease the number of joint-line incidences by $O(n^{2/3}k^{2/3} + n)$ at most. If we repeat this pruning step, as long as we can find a set L' as above of size $k > \sqrt{n'}$, where n' is the current number of lines that had survived the pruning, then this can be done at most $j = O(\sqrt{n})$ times before we have exhausted all lines. Since, for $k_1 + k_2 + \cdots + k_j \leq n$,

$$\sum_{i=1}^{j} k_i^{2/3} \leq j^{1/3} \left(\sum_{i=1}^{j} k_i \right)^{2/3} \leq j^{1/3} n^{2/3} = O(n^{5/6}) \ ,$$

we end up with a set of lines that has at most $O(j^{1/3}n^{4/3} + jn) = O(n^{3/2})$ fewer joint-line incidences than the original set. Consequently, as we are heading for a bound that is $\Omega(n^{3/2})$, we can *assume that every regulus or plane contains at most $\sqrt{n}$ lines of L.*

Good and Bad Triples. Let $\ell \in L$, and put $\mu(\ell) := |J_L \cap \ell|$. For every $p \in J_L \cap \ell$ there is a set $K_p \subseteq L_p \setminus \{\ell\}$ of two lines from L that form a non-coplanar triple with ℓ; we fix some $K_p = K_p(\ell)$ for every point $p \in \ell$ and set $L_\ell = \bigcup_{p \in J_L \cap \ell} K_p$. Clearly, $|L_\ell| = 2\mu(\ell)$. Consider an unordered triple $\{p_1, p_2, p_3\}$ of joints on ℓ. We call it *good*, if there are three pairwise skew lines $\ell_i \in K_{p_i}$, $i = 1, 2, 3$, and we call it *bad*, if there are three lines $\ell_i \in K_{p_i}$, $i = 1, 2, 3$, that lie in a common plane. We claim that any triple of joints on ℓ is good or bad.[5]

Put $K := K_{p_1} \cup K_{p_2} \cup K_{p_3}$. As it follows from the classical introductory example for the Ramsey Theory of graphs (see e.g. [3]) there must be a triple of lines in K that is either pairwise coplanar or pairwise skew. Since any two lines in K through the same point p_i are coplanar, a pairwise skew triple of lines in K must have one in each K_{p_i} and thus witnesses that the triple of joints at hand is good. If a triple of pairwise coplanar lines in K contains a pair through the same point p_i, then the third line forces this pair to be coplanar to ℓ, in contradiction to the choice of K_{p_i}. Hence, a pairwise coplanar triple of lines must come from distinct points on ℓ and it therefore

[5] In fact, it can be both—that's the way life goes.

certifies that the triple of joints is bad; (pairwise coplanar lines that intersect a common line in distinct points must lie in a common plane).

We fix some large constant $t \in \mathbb{N}$ and analyze triples of joints on lines $\ell \in L$, for which $\mu(\ell) > t\sqrt{n}$. Other lines can accumulate at most $O(n^{3/2})$ incidences with J_L. We cannot completely discard them, though, since that may invalidate some joints and thus many more joint-line incidences than those on such a 'light' line may get lost this way. However, we do not need to consider them among the 'base lines' ℓ in the ongoing analysis.

Good Triples on Lines with Many Joints. Fix a line $\ell \in L$ with $\mu = \mu(\ell) > t\sqrt{n}$. Every good triple $\{p_1, p_2, p_3\}$ of joints on ℓ has at least one triple ℓ_1, ℓ_2, ℓ_3 of pairwise skew lines, with $\ell_i \in K_{p_i}$, for $i = 1, 2, 3$. Put $\sigma = \sigma(\ell_1, \ell_2, \ell_3)$, and associate σ with $\{p_1, p_2, p_3\}$; note that $\ell \in \sigma$. Different choices of triples may generate the same regulus. In fact, a regulus ζ that contains k lines of L_ℓ may cause its complementary regulus $\zeta^\perp$ to be generated up to $\binom{k}{3}$ times.

Let $N_k = N_k(\ell)$ denote the number of reguli containing exactly k lines in L_ℓ, and let $N_{\geq k} = N_{\geq k}(\ell)$ denote the number of reguli with at least k lines in L_ℓ. Then, by construction,

$$T_G(\ell) \leq \sum_{k \geq 3} \binom{k}{3} N_k \ , \tag{10}$$

where $T_G(\ell)$ is the number of good triples on ℓ. In the next stages of the proof we show that (a) almost all triples in $J_L \cap \ell$ are good, and (b) the number of distinct reguli generated by the above process is $\Omega(\mu^3)$.

Claim 3.2 $N_{\geq k} = O\left(\dfrac{\mu^3}{k^5} + \dfrac{\mu}{k}\right)$.

Proof. Let R be any set of reguli that contain ℓ. Because of the first bound in (9) and the discussion following it, the number of pairs (ℓ', σ) with $\ell' \in L_\ell$, $\sigma \in R$, and $\ell' \in \sigma^\perp$ is at most $O(|L_\ell||R|^{2/3} + |R|)$. Next, in order to improve on this bound, we represent the lines in L_ℓ as points in an appropriate 3-dimensional parametric space (where a line $\lambda \in L_\ell$ is parametrized, e.g., by the point $\lambda \cap \ell$, and by its orientation, as a point on the unit sphere $\mathbb{S}^2$), and the reguli of R as curves in that space, where the curve corresponding to a regulus σ is the locus of all (points representing) lines that belong $\sigma^\perp$. The explicit form of such a curve can be easily worked out; under the above parametrization, these curves are algebraic curves of constant maximum degree. We then project this configuration onto some generic 2-plane. We obtain a set L_ℓ^* of 2μ points and a set R^* of $|R|$ curves in the plane. We wish to bound the number of incidences between these points and curves.

We choose some parameter r, to be determined below, and construct a $(1/r)$-*cutting* of the plane (see [5, 7]) into $O(r^2)$ pseudo-trapezoidal cells, each crossed by at most $|R|/r$ projected 'reguli-curves'. (Such a cutting exists, since the projections of the curves representing the reguli are algebraic of constant maximum degree; see, e.g., [7].) Moreover, by further splitting cells of the cutting, as necessary, we may also assume that each cell contains at most μ/r^2 projected points (lines of L_ℓ) in its interior. Applying the initial weaker bound, the number of incidences within the interior of a cell of the cutting is

$$O\left(\frac{\mu}{r^2}\left(\frac{|R|}{r}\right)^{2/3} + \frac{|R|}{r}\right) ,$$

which, summed over all cells, is

$$O\left(\frac{\mu|R|^{2/3}}{r^{2/3}} + |R|r\right) .$$

Using standard methods (as in [5]), one can show that this also bounds the number of incidences involving points that lie on the boundaries of the cells of the cutting. We now choose $r = \mu^{3/5}/|R|^{1/5}$. This number is between 1 and $|R|$, unless $\mu < |R|^{1/3}$ or $\mu > |R|^2$. In the former case, the weaker bound is $O(|R|)$. In the latter case, applying the second bound of (9), the number of incidences is $O(\mu)$. If r lies in the required range, the above bound becomes $O(\mu^{3/5}|R|^{4/5})$, so our incidence bound is always

$$O(\mu^{3/5}|R|^{4/5} + \mu + |R|) .$$

Apply this bound to the set R of reguli with at least k generating lines in L_ℓ. Since the number of incidences $(\ell', \sigma^\perp)$, with $\ell' \in L_\ell$, $\sigma \in R$, is at least $k|R|$ in this case, it follows that

$$k|R| = O(\mu^{3/5}|R|^{4/5} + \mu + |R|) ,$$

from which the bound on $N_{\geq k} = |R|$ follows readily. $\square$

We can now derive a bound for the following partial sum (where we clearly have $t < \mu^{1/2}$)

$$\begin{aligned}\sum_{k\geq t}\binom{k}{3}N_k &= \binom{t}{3}N_{\geq t}+\sum_{k>t}\binom{k-1}{2}N_{\geq k}\\ &= O\left(\frac{\mu^3}{t^2}+\sum_{k=t}^{\lfloor\mu^{1/2}\rfloor}\frac{\mu^3}{k^3}+\sum_{k=\lfloor\mu^{1/2}\rfloor+1}^{k_{\max}}\mu k\right) ,\end{aligned}$$

where $k_{\max}$ is the maximum number of lines of L_ℓ that appear in a single complementary regulus. By our pruning assumption, we have $k_{\max} \leq \sqrt{n} < \mu/t$ and so

$$\sum_{k\geq t}\binom{k}{3}N_k = O\left(\frac{\mu^3}{t^2}+\mu k_{\max}^2\right) = O\left(\frac{\mu^3}{t^2}\right) .$$

Recall (10) to conclude

$$T_G(\ell) = \sum_{k<t}\binom{k}{3}N_k + O\left(\frac{\mu^3}{t^2}\right) . \tag{11}$$

There are Very Few Bad Triples. Let $T_B(\ell)$ denote the number of bad triples in $J_L \cap \ell$. Let $\{p_1, p_2, p_3\}$ be such a bad triple; there is a plane π that contains lines $\ell_i \in K_{p_i}$, for $i = 1, 2, 3$. Different choices of triples $\{p_1, p_2, p_3\}$ may yield in this manner the same plane. In fact, if π is a plane containing k lines of L_ℓ, it may be encountered up to $\binom{k}{3}$ times.

Let $M_k = M_k(\ell)$ denote the number of planes containing exactly k lines in L_ℓ, and let $M_{\geq k} = M_{\geq k}(\ell)$ denote the number of planes with at least k lines in L_ℓ. Since no two planes in question have a common line other than ℓ itself, it follows that $M_{\geq k} \leq 2\mu/k$. Then, by construction,

$$\begin{aligned}T_B(\ell) &\leq \sum_{k\geq 3}\binom{k}{3}M_k = \sum_{k\geq 3}\binom{k-1}{2}M_{\geq k}\\ &= O\left(\sum_{k=3}^{\tilde{k}_{\max}}k\mu\right) = O(\mu\tilde{k}_{\max}^2) ,\end{aligned}$$

where $\tilde{k}_{\max} \leq \sqrt{n} < \mu/t$ is the maximum number of lines of L_ℓ on any single plane. Hence, $T_B(\ell) = O(\mu^3/t^2)$. So if t is chosen to be a sufficiently large constant, it follows from $T_B(\ell) + T_G(\ell) \geq \binom{\mu}{3}$ that

$$T_G(\ell) = \Omega(\mu^3) \ . \tag{12}$$

The Final Stage of the Proof. The bounds (12) and (11) imply that

$$\mu^3 = O(t^3) \sum_{k<t} N_k \ .$$

Since t is a constant, the number of distinct reguli generated by triples of points along ℓ, as above, is $\Omega(\mu^3)$.

Let L' denote the set of all lines $\ell \in L$ with $\mu(\ell) > t\sqrt{n}$, and let R be the set of all reguli containing at least 3 lines of L. Clearly, $|R| = O(n^3)$ and, for every $\ell \in L'$, there are $\Omega(\mu(\ell)^3)$ distinct reguli σ in R with $\ell \in \sigma$.

On the one hand, from what we have just shown, it follows that $\Omega\left(\sum_{\ell \in L'} \mu(\ell)^3\right)$ is a lower bound for the number of pairs (ℓ', σ) with $\ell' \in L'$, $\sigma \in R$, and $\ell' \in \sigma$. On the other hand, the forbidden subgraph argument (see (9)) yields an upper bound of $O(|L'||R|^{2/3} + |R|) = O(n^3)$ for the number of such pairs. Now, using Hölder's inequality,

$$\sum_{\ell \in L'} \mu(\ell) \leq \left(\sum_{\ell \in L'} \mu(\ell)^3 \right)^{1/3} \cdot |L'|^{2/3} = O(n^{5/3}),$$

thus completing the proof of the theorem. $\Box$

Immediate Implications. It is interesting to note that one can 'distill' from the preceding proof the following result, reminiscent of Beck's theorem for lines in the plane [2].

Corollary 3.3 *Let ℓ be a fixed line in 3-space, and let L be a set of n lines incident to ℓ, which meet ℓ at $n/2$ distinct points, such that each of these points is incident to two lines of L, and such that any such pair of lines do not form a coplanar triple with ℓ. Then there exists a constant $c > 0$ such that one of the following properties hold:*

(i) There exists a regulus generated by at least cn lines of L.

(ii) There exists a plane containing at least cn lines of L.

(iii) There exist at least cn^3 distinct reguli generated by the lines of L.

Returning to our measures of incidences, we have:

Corollary 3.4 *For any sets P of m points and L of n lines in $\mathbb{R}^3$, $I_c(P, L) = m + O(n^{5/3})$ and $I_\nu(P, L) = m + O(n^{5/3})$.*

Proof. Each point p with $c_p(L) = 1$ contributes 1 to $I_c(P, L)$. The set P' of remaining points are joints of L. Since we have $I_c(P', L) \leq I(P', L)$, the corollary follows for I_c, and, similarly, also for I_ν. $\Box$

Clearly, we can also conclude that $|J_L| = O(n^{5/3})$, which improves a bound of $O(n^{7/4})$ obtained in [4], but is inferior to a later improvement in [12] to $O(n^{23/14} \log^{31/14} n) = O(n^{1.643})$. It is a challenging open problem to further improve this bound on the number of joints, probably close to $O(n^{3/2})$, which is the best known lower bound for the number of joints.

4 Plane-Cover Number

Theorem 4.1 *For any sets P of m points and L of n lines in $\mathbb{R}^3$, $I_c(P, L) = O(m^{4/7} n^{5/7} + m + n)$.*

The bound is linear in m for $m > n^{5/3}$, in agreement with Corollary 3.4. For smaller values of m, we improve the bound using the following partitioning scheme.

Project the points of P and the lines of L onto some generic plane π, and dualize the configuration within π, mapping points to lines and lines to points, while preserving incidences between them. We thus get a set P^* of m lines and a set L^* of n points. Construct a $(1/r)$-cutting of that dual plane (see [5]), for some parameter r to be fixed later. We obtain $O(r^2)$ cells, each crossed by at most m/r lines of P^*. By further cutting the cells into subcells, as necessary, we may also assume that the interior of each cell contains at most $O(n/r^2)$ points of L^* (while their number remains $O(r^2)$).

A crucial property of our $I_c(P, L)$ is that it is *subadditive* under partitioning. That is, if $L = L_1 \cup L_2$ then $c_p(L) \leq c_p(L_1) + c_p(L_2)$, which follows trivially by definition, and hence

$$I_c(P, L) \leq I_c(P, L_1) + I_c(P, L_2).$$

Hence we obtain, taking also into account the $O(mr)$ real incidences that involve lines whose dual points lie on cell boundaries (as follows from the analysis in [5]),

$$
\begin{aligned}
I_c(P, L) &= O(mr) + O(r^2) \cdot O\left(\frac{m}{r} + \left(\frac{n}{r^2}\right)^{5/3}\right) \\
&= O\left(mr + \frac{n^{5/3}}{r^{4/3}}\right).
\end{aligned}
$$

We choose $r = n^{5/7}/m^{3/7}$ to balance the two terms. We have to make sure that $1 \le r \le m$, that is, $n^{1/2} \le m \le n^{5/3}$. If $m > n^{5/3}$ then we have $I_c(P, L) = O(m)$. If $m < n^{1/2}$ then, using the Szemerédi-Trotter bound on the number of incidences between points and lines in the plane (which also holds for points and lines in three dimensions), we have $I_c(P, L) \le I(P, L) = O(n)$. Otherwise, r lies in the required range, and we get $I_c(P, L) = O(m^{4/7}n^{5/7})$. Hence, putting together the various bounds, the assertion of Theorem 4.1 follows. $\Box$

Remark. Corollary 3.4 (and Theorem 3.1) bound the number of real incidences involving the joints of L, but this is not the case for Theorem 4.1. The reason is that, in the proof of this theorem, a point p in the set J_L may lose, under the partitioning scheme used there, the property that its incident lines are not all coplanar (that is, of being a joint). To make this remark more concrete, consider the following construction. Let P be a set of n points in the xy-plane, and let L_0 be a set of n lines in that plane, such that the number of (real) incidences between L_0 and P is $\Theta(n^{4/3})$ (see,. e.g., [6]). Let L_1 be an additional set of z-vertical lines passing through the points of P, and put $L = L_0 \cup L_1$. Each point of P is a joint of L. However, the bound in Theorem 4.1, namely $O(n^{4/7} \cdot n^{5/7}) = O(n^{9/7}) \ll \Theta(n^{4/3})$, does not indeed apply to the real incidence count, for the reason stated above. (As a matter of fact, for these sets of points and lines, we have $I_c(P, L) = O(n)$, as is easily verified.)

5 Equally Inclined Lines

Theorem 5.1 *For a given angle $\theta \in (0, \pi/2)$, let L be a set of n lines in $\mathbb{R}^3$, each forming an angle of θ with the xy-plane. Let P be a set of m points in $\mathbb{R}^3$. Then $I(P, L) = O(\min\{m^{3/4}n^{1/2}\kappa(m), m^{4/7}n^{5/7}\} + m + n)$, where $\kappa(m) = (\log m)^{O(\alpha^2(m))}$, and $\alpha(m)$ is the slowly growing inverse Ackermann function.*

The bound of $O(m^{4/7}n^{5/7} + m + n)$ is taken care of by Theorem 4.1, because no three lines of L passing through a common point can be coplanar, so we can confine ourselves to the derivation of the other bound. By an appropriate scaling of the z-axis, we may assume that $\theta = \pi/4$. We map the given configuration into the xy-plane as follows. A line $\ell \in L$ is projected vertically to a line ℓ^* in the xy-plane, on which we mark the point $w(\ell)$ of intersection between ℓ and the plane. A point $p = (a, b, c) \in P$ is mapped to the circle p^* with center (a, b) and radius c. See Figure 1.

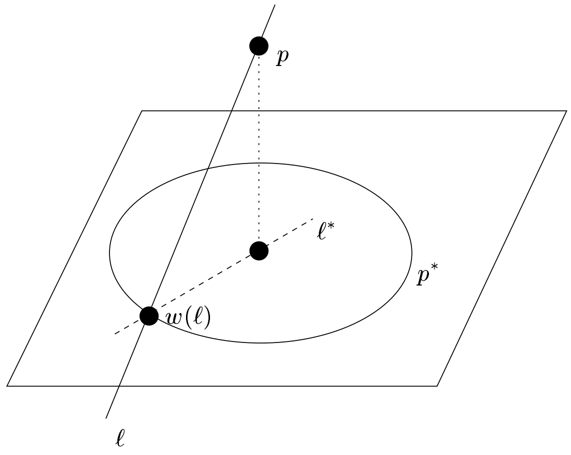

Figure 1: Mapping points and equally-inclined lines to the xy-plane.

Note that p lies on ℓ if and only if the following two conditions are satisfied:

(i) The line ℓ^* passes through the center of the circle p^*.

(ii) The circle p^* passes through the point $w(\ell)$.

Note that in this case ℓ^* and p^* are orthogonal to each other. In particular, if the line $\ell \in L$ is incident to points $p_1, p_2, \ldots, p_s$ then the circles $p_1^*, p_2^*, \ldots, p_s^*$ are all tangent to each other at the common point $w(\ell)$.

Let I denote the number of incidences between P and L. We may assume that each line of L is incident to at least $t := I/(an)$ points, for some constant $a \geq 2$, because all the other lines contribute less than I/a to the total incidence count, and we may simply ignore them.

We have thus reduced the problem to either of the two following problems:

Problem 5.2 *Let C be a given finite set of circles and Q a finite set of points in the plane, where each point is equipped with a direction. Let $T(C, Q)$ be the number of pairs (c, q), for $c \in C$, $q \in Q$, such that q lies on c and c is orthogonal to q's direction (so, as above, all circles that form such pairs with the same point q are tangent to each other at q).*

(i) Obtain a bound for $T(C, Q)$.

(ii) Given C as above and an integer parameter k, obtain an upper bound for $T(C, Q_{\geq k})$, where $Q_{\geq k}$ is the set of all points at which at least k circles of C are tangent to each other (with the direction associated with such a point being the radial direction of all tangent circles).

We prefer to tackle Problem 5.2(ii); as will be shown, this is sufficient for obtaining the bound of Theorem 5.1. Let C be a set of m circles in the plane. A *pencil* π of *weight* $\omega_\pi = j$ corresponds to a point u, and a direction d, so that exactly j circles of C pass through u and are orthogonal to d at u (so they are all tangent to each other at u). The pencil itself is the collection of these circles.

We seek an upper bound on the sum of the weights of any set Π of pencils, for which $\omega_\pi \geq k$ for each $\pi \in \Pi$. We obtain this bound as follows.

Let P be the set of points in $\mathbb{R}^3$ that are mapped to the circles of C; $|C| = |P| = m$. Project the points of P on some generic plane h, so that no two points are projected to the same point. Apply to the projected set $\tilde{P}$ the partitioning theorem of Matoušek [8], which, for a given parameter $r \leq m$ (that will be specified shortly), yields a partitioning of $\tilde{P}$ into $q = O(r)$ subsets, call them $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_q$, each consisting of at least two points and at most m/r points, so that no line (in the plane of projection h) crosses (i.e., intersects but does not fully contain) the convex hulls of more than $cr^{1/2}$ sets, for some absolute constant $c > 0$. Let C_i denote the subset of C consisting of circles whose representing points are projected to points in $\tilde{P}_i$, for $i = 1, 2, \ldots, q$.

As shown by Aronov and Sharir [1], and later slightly improved in [9], a set of N circles in the plane can be cut into $O(N^{3/2}\kappa(N))$ arcs, where $\kappa(N)$ is as defined above, so that each pair of arcs intersect at most once. We adapt the analysis of [1] to the case at hand, which is highly degenerate due to the multitude of tangencies. Since a tangency is a degenerate case of a double intersection, it follows from the analysis of [1, 9] that if ξ circles are tangent to each other at some common point q, then the cutting procedure will have to make at least $\xi - 1$ cuts of the circles at q. This can be seen either by specializing the analysis of [1, 9] to the case of tangencies, or by applying a random small perturbation to the circles, thereby turning each multiple tangency into a small region, inside which many pairs of the formerly tangent circles have double intersections; in that case, it follows that the number of cuts that would be required within each such small region have to be proportional to the number of circles that participate in the tangency.

We apply this cutting to each subset C_i, thereby cutting the circles of C into

$$O(r) \cdot O((m/r)^{3/2}\kappa(m/r)) = O(m^{3/2}\kappa(m)/r^{1/2})$$

arcs, so that any two arcs, that come from circles in the same subset C_i, intersect at most once, and their relative interiors are not tangent to each other.

Consider a pencil $\pi \in \Pi$ consisting of $j \geq k$ circles. Let C_i be a subset that contains at least two circles of π. Then, as just argued, all the circles of π that belong to C_i, with the possible exception of at most one circle, will be cut by the above process at the pencil point.

Consider next those subsets C_i that contain only one arc of π. The points that represent these circles in 3-space all lie on the line ℓ that projects onto the axis of the pencil, passes through the point of the pencil, and forms an angle of θ degrees with the xy-plane. The projection $\tilde{\ell}$ of ℓ onto the plane h meets the convex hulls of the corresponding sets $\tilde{P}_i$. If $\tilde{\ell}$ fully contains the convex hull of such a $\tilde{P}_i$, then all circles corresponding to points in this set belong to the pencil, implying that $|\tilde{P}_i| = 1$, contrary to the properties of the construction. Hence, $\tilde{\ell}$ properly crosses the convex hull of each of these sets, so the number of such sets, and thus their overall contribution to the size of the pencil, is at most $cr^{1/2}$.

We choose r so that $cr^{1/2} = k/2$. This implies that the total number of circles of π is at most 4 times larger than the number of circles that have been cut by the above process at the pencil point. Summing this over all pencils of Π, and recalling that no cutting point on a circle is 'charged' more than once, we conclude that the overall weight of the pencils in Π is at most

$$O(m^{3/2}\kappa(m)/r^{1/2}) = O(m^{3/2}\kappa(m)/k) \ .$$

We now apply this bound in our setup, with $k = t = I/(a|L|)$. We note that the above partition makes sense only when $1 \leq r \leq m = |P|$, i.e., when $2c \leq k \leq 2c\sqrt{|P|}$, or when $2ac|L| \leq I \leq 2ac\sqrt{|P|}|L|$. Note that the incidence graph between C and $Q_{\geq k}$ does not contain any $K_{2,2}$ as a subgraph, so, by the extremal graph theoretic bounds [10] that we have already exploited, we always have $I = O(|P|^{1/2}|L| + |P|)$. If $|L| \leq \sqrt{|P|}$ then $I = O(|P|)$, so we may assume that $|L| \geq \sqrt{|P|}$, in which case $I = O(|P|^{1/2}|L|)$. In other words, either $I = O(|P| + |L|)$ or, by appropriately choosing the constant a in the definition of k, we may assume that k lies in the required range. In this case, we obtain the inequality

$$I = \frac{O(|P|^{3/2}\kappa(|P|)|L|)}{I} ,$$

from which we obtain[6]

$$I = O(|P|^{3/4}|L|^{1/2}\kappa(|P|)).$$

Taking into account also the case $I = O(|P| + |L|)$, we thus obtain

$$I(P, L) = O(|P|^{3/4}|L|^{1/2}\kappa(|P|) + |P| + |L|) .$$

Remark. It has been conjectured in [1, 9] that the number of cuts needed to eliminate all tangencies (or, more generally, all 'lenses') in a collection C of circles in the plane is only $O(|C|^{4/3})$. Any improvement towards this bound would of course lead to a similar improvement in Theorem 5.1.

6 An Alternative Incidence Count

Recall the definition of the alternative measure of noncoplanar incidences, via the function $\nu_p(L) = \sqrt{s_p}$, where s_p is the number of *all* planes spanned by at least two lines of L_p. In this section we analyze ν_p and its relation to c_p. We note two properties of ν_p: (i) ν_p majorizes c_p, and (ii) ν_p is *not* subadditive, which makes it unsuitable for analysis that uses any partitioning scheme, as the one used in the proof of Theorem 4.1. Nevertheless, the two measures yield the same weaker asymptotic bound, given in Theorem 3.1 and Corollary 3.4.

Lemma 6.1 $c_p(L) = O(\nu_p(L))$.

[6] Here the constant in the exponent of the expression for $\kappa(\cdot)$ is halved, but the asymptotic form of the expression remains the same.

Proof. Fix p and consider the set L_p of lines of L incident to p. By slicing L_p by some generic plane not passing through p, the problem reduces to the following: Given a set S of points in the plane, let $c(S)$ denote the minimum number of lines that cover S, and let $\nu(S)$ denote the square root of the number of distinct lines spanned by at least two points of S. We need to show that $c(S) = O(\nu(S))$.

If all points of S are collinear then $c(S) = \nu(S) = 1$, so assume that S is not contained in a single line.

We use Beck's theorem [2]: There is an absolute constant $b > 0$ such that if no line contains more than $b|S|$ points of S then the number of lines spanned by S is at least $b|S|^2$. In the latter case, $\nu(S) \geq \sqrt{b}|S|$. On the other hand, if there is a line ℓ containing at least $b|S|$ points of S, and not all points of S lie on ℓ, then there exist at least $b|S|$ distinct lines spanned by S—those connecting the points on ℓ to a single point not lying on ℓ. In this case, $\nu(S) \geq \sqrt{b}|S|^{1/2}$.

Suppose that there exists a line ℓ containing at least $b|S|$ points of S. Remove $b|S|$ of these points from S, and use ℓ as one of the covering lines in the definition of $c(S)$. Let S' be the set of remaining points. If there exists a line ℓ' containing at least $b|S'|$ points of S', remove these points, add ℓ' to the set of covering lines, and keep repeating this step as long as possible, Suppose j such steps are executed, and let $S^{(j)}$ denote the set of remaining points. Note that $j = O(\log |S|)$. We have

$$c(S) \leq j + c(S^{(j)}) \ .$$

If $|S^{(j)}| = O(\log |S|)$ then

$$c(S) = O(\log |S|) = O(|S|^{1/2}) = O(\nu(S)) \ .$$

Otherwise, by Beck's Theorem, $\nu(S) \geq \nu(S^{(j)}) = \Omega(|S^{(j)}|)$. Hence,

$$\begin{aligned} c(S) &\leq O(\log |S|) + |S^{(j)}| = O(|S^{(j)}|) \\ &= O(\nu(S^{(j)})) = O(\nu(S)) \ . \end{aligned}$$

□

REMARK. It does not always hold that $\nu_p(L \cup L') \leq \nu_p(L) + \nu_p(L')$.

Indeed, suppose that all the lines of L lie in one common plane π, and all the lines of L' lie in another common plane π'. Then $\nu_p(L) + \nu_p(L') = 2$, but $\nu_p(L \cup L') \approx \sqrt{|L| \cdot |L'|}$.

7 Lower Bounds

Theorem 7.1 *For $m, n \in \mathbb{N}$ with $m^{2/3} \leq n \leq m^{4/3}$, there are sets P of m points and L of n equally inclined lines in $\mathbb{R}^3$, such that $I(P, L) = \Omega(m^{2/3} n^{1/2})$.*

Given m, we let P be the set of integer grid points in $[-M, M]^3$, with $M := (\lfloor \sqrt[3]{m} \rfloor - 1)/2 = \Theta(\sqrt[3]{m})$ ensuring $|P| \leq m$.

We consider lines at directions v that form an angle of $\theta := \arccos(1/\sqrt{2}) = \pi/4$ with the z-axis, i.e. for $(a, b, c) = v$,

$$\frac{(a, b, c)(0, 0, 1)}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{2}} \iff a^2 + b^2 = c^2$$

We wish to choose an appropriate set of such lines incurring many incidences with P. For integers ξ, η with $(\xi, \eta) = 1$ (i.e. relatively prime) and $1 \leq \eta \leq \xi \leq \sqrt{M}$, we define the set $L(\xi, \eta)$ of lines that pass through a point in $P' := P \cap [-M/2, M/2]^3$ at direction $(\xi^2 - \eta^2, 2\xi\eta, \xi^2 + \eta^2)$. Each of these lines has angle θ with the z-axis. Since each point in P' infers exactly one incidence with $L(\xi, \eta)$, and each point in P at most one, the overall number of incidences between $L(\xi, \eta)$ and P is $\Theta(M^3)$. Given this and the fact that each line in $L(\xi, \eta)$ passes through $\Theta(M/(\xi^2 + \eta^2))$ points in P, we know that $|L(\xi, \eta)|$ has to be $\Theta(M^2(\xi^2 + \eta^2))$. Let now

$$L := \bigcup_{(\xi,\eta)=1;\, 1 \leq \eta \leq \xi \leq t} L(\xi, \eta)$$

for a parameter t, $1 \leq t \leq \sqrt{M}$, that we will put momentarily in order to see to it that $|L| \leq n$. Recall that the number of positive integers at most ξ relatively prime to ξ is denoted by Euler's function $\varphi(\xi)$. We have

$$I(P, L) = \Theta(M^3) \cdot \sum_{\xi=1}^{t} \varphi(\xi) = \Theta(M^3 t^2) = \Theta(m t^2)$$

and

$$\begin{aligned} |L| &= \sum_{\xi=1}^{t} \sum_{(\xi,\eta)=1;\, 1 \leq \eta \leq \xi} \Theta(M^2(\xi^2 + \eta^2)) \\ &= \sum_{\xi=1}^{t} \varphi(\xi) \Theta(M^2 \xi^2) = \Theta(M^2 t^4) = \Theta(m^{2/3} t^4) \; . \end{aligned}$$

Hence we can take t of the order $n^{1/4}/m^{1/6}$ as long as $1 \leq t \leq \sqrt{M}$; this is possible in the asserted range of $m^{2/3} \leq n \leq m^{4/3}$. Then this gives $\Theta(mt^2) = \Theta(m^{2/3}n^{1/2})$ incidences and the bound claimed in Theorem 7.1 follows. $\Box$

Note that this bound is indeed superlinear when $m^{2/3} \ll n \ll m^{4/3}$.

Theorem 7.2 *For $m, n \in \mathbb{N}$ with $m^{2/3} \leq n \leq m^2$, there are sets P of m points and L of n lines in $\mathbb{R}^3$, such that $I_\nu(P, L) = \Omega(m^{1/2}n^{3/4})$.*

Let m and n be as in the theorem. Construct a set P of m points and a set L of n lines in 3-space. We take P to be the set of points of the integer lattice whose coordinates are all between 1 and $m^{1/3}$. The lines of L are constructed as follows. We fix an integer parameter t, and for each $r = 1, 2, \ldots, t$, we consider the collection $L^{(r)}$ of the lines

$$y = \frac{\xi}{r}(x - x_0) + y_0, \quad z = \frac{\eta}{r}(x - x_0) + z_0,$$

where $1 \leq \xi, \eta < r$ are both relatively prime to r, $1 \leq x_0 \leq r$ and $1 \leq y_0, z_0 \leq \frac{1}{2}m^{1/3}$. We take $L = \bigcup_{r \leq t} L^{(r)}$. The number of lines in L is

$$\frac{m^{2/3}}{4} \sum_{r=1}^{t} r\varphi^2(r) = \Theta(m^{2/3}t^4) \ .$$

Hence, if we choose t to be proportional to $n^{1/4}/m^{1/6}$, we can make the size of L equal to n.

The number of incidences between the lines of L and the points of P is

$$\begin{aligned} I(P, L) &= \frac{m^{2/3}}{4} \sum_{r=1}^{t} r\varphi^2(r) \cdot \left\lfloor \frac{m^{1/3}}{2r} \right\rfloor = \Theta(m) \sum_{r=1}^{t} \varphi^2(r) \\ &= \Theta(mt^3) = \Theta(m^{1/2}n^{3/4}) \ . \end{aligned}$$

For this construction to make sense, we need $t = n^{1/4}/m^{1/6}$ to be between 1 and $m^{1/3}$; that is, we should have $n^{1/2} \leq m \leq n^{3/2}$, which, indeed, is the assumed range. We have thus obtained a set P of m points and a set L of n lines that satisfy $I(P, L) = \Omega(m^{1/2}n^{3/4})$. We next show that this is also a lower bound for $I_\nu(P, L)$.

Let $p \in P$, and let L_p denote the set of lines incident to p; without loss of generality, assume that p is the origin. It suffices to show that the number of distinct planes spanned by pairs of lines in L_p is $\Omega(|L_p|^2)$.

By intersecting the lines of L_p with the plane $\pi: \ x = 1$, we obtain a collection S_p of $M = |L_p|$ points in π, so that

$$S_p = \{(1, \xi/r, \eta/r) \ \mid \ 1 \le r \le t, \ 0 \le \xi, \eta < r, \\ (\xi, r) = (\eta, r) = 1\} \ .$$

Our goal is to show that the number of distinct lines that they determine is $\Omega(M^2)$. Let Λ denote the collection of these lines.

Lemma 7.3 *The maximum number of points of S_p on a line in Λ is $O(M^{2/3})$.*

Proof. Let $\lambda \in \Lambda$, let h be the plane spanned by p and λ, and let h^+ denote its unit normal vector. Each line $\ell \in L_p$ that passes through a point in $S_p \cap \lambda$ is orthogonal to h^+. If ℓ is represented by the triple (ξ, η, r), then we have $h^+ \cdot (r, \xi, \eta) = 0$. In other words, each point of $S_p \cap \lambda$ can be represented by a point of the 3-dimensional integer lattice that lies in a fixed plane Q (through the origina) and has L_∞-norm at most t.

We claim that the number of such points is $O(t^2)$. Indeed, denote the set of those representing points by D. For each point $w \in D$, let σ_w be a ball of radius $1/2$ centered at w. Clearly, all these balls have pairwise-disjoint interiors and they are all contained in a cylinder whose height is 1 and whose base is a disk of radius $t\sqrt{3}$. Hence the size of D is at most

$$\frac{3\pi t^2}{\pi/6} = O(t^2) = O(M^{2/3}),$$

as claimed. $\Box$

Theorem 7.2 is now an immediate consequence of Beck's theorem (as stated in the proof of Lemma 6.1 above), applied to S_p. $\Box$

Remark. Note that, in the range $m \le n^{3/2}$, assumed in both theorems, the bound in Theorem 7.2 is larger than that in Theorem 7.1. Unfortunately, in the construction in the proof of Theorem 7.2, $c_p(L)$ is much smaller than $|L_p|$: Since S_p is the union of (subsets of) t grids, each of size at most $t \times t$, it can be covered by $O(t^2)$ lines. Hence, for this construction, we have

$$I_c(P, L) = O(mt^2) = O(n^{1/2}m^{2/3}),$$

which is the same as the lower bound of Theorem 7.1 for the number of incidences between points and equally-inclined lines.

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