Stability of the moment theorem

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A very general and frequently used theorem in discrete geometry is the "moment theorem" of L. Fejes Tóth. It states the following:

Let H be a convex polygon in E^2 with at most six sides and f a nondecreasing function defined for non negative reals. Then for any set S of n points we have

$$\int_H \min_{p\in S} f(||x-p||)dx \geq n\int_{H_n} f(||x||)dx,$$

where H_n is a regular hexagon of area $|H|/n$ centered at the origin $||\cdot||$ denotes Euclidean norm and $|\cdot|$ denotes Lebesgue measure in E^2.

Recently, P. Gruber announced a weak extension of this theorem to Riemannian 2-manifolds as well as a stability version of it in the case when $F(t) = t^\alpha$ with $0 < \alpha \leq 2$. We give a proof which is valid for all strictly increasing functions.