Kepler and the Sausage

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Abstract

The density of finite sphere packings $C_n + B^d$ in Euclidean d–space E^d with $C_n = \{c_n, \ldots, c_n\}$ is measured by parametric density

$$\delta(B^d, C_n, \rho) = n\kappa_d / V(\mathrm{conv} C_n + \rho B^d)$$

V: volume, $V(B^d) = \kappa_d$ and $\rho > 0$. Any C_n, for which $\min V(\mathrm{conv} C_n + \rho B^d)$ is attained, is called a densest packing configuration and is denoted by $C_{n,\rho}$ and conv $C_{n,\rho}$ by $P_{n,\rho}$.

The following conjecture indicates that there are critical parameters, where the shape of $P_{n,\rho}$ changes extremely (sausage catastrophe).

Conjecture: "Dense sphere packings have extreme dimension",
or more precisely: For any $d \geq 2$ and $n \geq d+1$ there is a $\rho_{d,n} > 0$ with

a) dim $P_{n,\rho} = 1$ (sausage) for $\rho \leq \rho_{d,n}$

b) dim $P_{n,\rho} = d$ (cluster) for $\rho > \rho_{d,n}$

and $I(P_{n,\rho}) \geq I(T^d)$, where I denotes the isoperimetric quotient and T^d the regular d–simplex.

For $d = 2$ this is true (Groemer 1960). For $d \geq 3$ this conjecture is supported by partial results of various authors. A consequence of this conjecture is

$$\limsup_{n\to\infty} (\rho_{d,n})^{d-1} = \delta_d / \delta_s$$

where δ_d is the density of the classical densest sphere packing (Kepler) and $\delta_s = \kappa_d / (2\kappa_{d-1})$ the density of the (infinite) sausage packing.